NELSON SENIOR MATHS METHODS 12

FULLY WORKED SOLUTIONS

CHAPTER 1: Derivatives, Exponential and trigonometric functions

Exercise 1.01 The product and quotient rules

Concepts and techniques

1 **a** $\frac{d}{dx}[x^3(2x+3)] = 3x^2(2x+3) + 2x^3$

$= 8x^3 + 9x^2$

NOTE: These derivatives can be expressed in several ways.

A couple are listed below.

$[x^3(2x+3)] = 3x^2(2x+3) + 2x^3$

$= 8x^3 + 9x^2$

OR Let $y = [x^3(2x+3)]$

$\frac{dy}{dx} = 3x^2(2x+3) + 2x^3$

$= 8x^3 + 9x^2$

b $\frac{d}{dx}(3x-2)(2x+1) = 3(2x+1)+2(3x-2)$

$= 12x - 1$

c $\frac{d}{dx}3x(5x+7) = 3(5x+7) + 5(3x)$

$= 30x + 21$

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d $\dfrac{d}{dx}4x^4(3x^2 - 1) = 16x^3(3x^2 - 1) + 6x(4x^4)$

$= 72x^5 - 16x^3$

e $\dfrac{d}{dx}2x(3x^4 - x) = 2(3x^4 - x) + (12x^3 - 1)2x$

$= 30\ x^4 - 4x$

2 **a** $\dfrac{d}{dx}\ (3x - 7)(2x + 5) = 3(2x + 5) + 2(3x - 7)$

$= 12x + 1$

b $\dfrac{d}{dx}\ (5x + 2)(x^2 - 1) = 5(x^2 - 1) + 2x(5x + 2)$

$= 15x^2 + 4x - 5$

c $\dfrac{d}{dx}\ (5x - 3)(x^3 + 2x - 1) = 5(x^3 + 2x - 1) + (3x^2 + 2)\ (5x - 3)$

$= 5x^3 + 10x - 5 + 15x^3 - 9x^2 + 10x - 6$

$= 20x^3 - 9x^2 + 20x - 11$

d $\dfrac{d}{dx}\ (x^2 + 7)(x^2 - x - 1) = 2x(x^2 - x - 1) + (2x - 1)(x^2 + 7)$

$= 2x^3 - 2x^2 - 2 + 2x^3 - x^2 + 14x - 7$

$= 4x^3 - 3x^2 + 14x - 9$

e $\dfrac{d}{dx}\ (x^3 + 3)(x^4 + 2x^3 - 5x^2 + x - 2)$

$= 3x^2(x^4 + 2x^3 - 5x^2 + x - 2) + (4x^3 + 6x^2 - 10x + 1)\ (x^3 + 3)$

$= 3x^6 + 6x^5 - 15x^4 + 3x^3 - 2x^2 + 12x^6 + 6x^5 - 10x^4 + x^3 + 12x^3 + 18x^2 - 30x + 3$

$= 18x^6 + 12x^5 - 9x^4 + x^3 + 16\ x^3 + 16x^2 - 30x + 3$

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3 **a** $$\frac{d}{dx}\left(\frac{1}{2x-1}\right)=\frac{0\left(2x-1\right)-2(1)}{\left(2x-1\right)^2}=\frac{-2}{\left(2x-1\right)^2}$$

b $$\frac{d}{dx}\left(\frac{3x}{x+5}\right)=\frac{3\left(x+5\right)-1(3x)}{\left(x+5\right)^2}=\frac{15}{\left(x+5\right)^2}$$

c $$\frac{d}{dx}\left(\frac{x^3}{x^2-4}\right)=\frac{3x^2\left(x^2-4\right)-(2x)x^3}{\left(x^2-4\right)^2}=\frac{x^4-12x^2}{\left(x^2-4\right)^2}$$

d $$\frac{d}{dx}\left(\frac{x-3}{5x+1}\right)=\frac{1\left(5x+1\right)-5(x-3)}{\left(5x+1\right)^2}=\frac{16}{\left(5x+1\right)^2}$$

e $$\frac{d}{dx}\left(\frac{x-7}{x^2}\right)=\frac{x^2-2x(x-7)}{x^4}=\frac{-x^2+14x}{x^4}=\frac{-x\ \ +14}{x^3}$$

f $$\frac{d}{dx}\left(\frac{5x+4}{x+3}\right)=\frac{5(x+3)-1\left(5x+4\right)}{\left(x+3\right)^2}=\frac{11}{\left(x+3\right)^2}$$

g $$\frac{d}{dx}\left(\frac{x}{2x^2-x}\right)=\frac{1\left(2x^2-x\right)-(4x-1)x}{\left(2x^2-x\right)^2}=\frac{-2x^2}{\left(2x^2-x\right)^2}$$

h $$\frac{d}{dx}\left(\frac{x+4}{x-2}\right)=\frac{1\left(x-2\right)-1(x+4)}{\left(x-2\right)^2}=\frac{-6}{\left(x-2\right)^2}$$

i $$\frac{d}{dx}\left(\frac{2x+7}{4x-3}\right)=\frac{2(4x-3)-4\left(2x+7\right)}{\left(4x-3\right)^2}=\frac{-34}{\left(4x-3\right)^2}$$

j $$\frac{d}{dx}\left(\frac{x+5}{3x+1}\right)=\frac{1\left(3x+1\right)-3(x+5)}{\left(3x+1\right)^2}=\frac{-14}{\left(3x+1\right)^2}$$

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4 **a** $\frac{d}{dx}(x^{-4}) = -4x^{-5}$

b $\frac{d}{dx}(x^{-8}) = -8x^{-9}$

c $\frac{d}{dx}(2x^{-3}) = -6x^{-4}$

d $\frac{d}{dx}(5x^{-11}) = -55x^{-12}$

e $\frac{d}{dx}\left(\frac{x^{-9}}{5}\right) = \frac{-9}{5}x^{-10}$

f $\frac{d}{dx}\left(x^{\frac{1}{2}}\right) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2x^{\frac{1}{2}}}$

g $\frac{d}{dx}\left(x^{\frac{1}{4}}\right) = \frac{1}{4}x^{-\frac{3}{4}} = \frac{1}{4x^{\frac{3}{4}}}$

h $\frac{d}{dx}\left(3x^{\frac{1}{7}}\right) = \frac{3}{7}x^{-\frac{6}{7}} = \frac{3}{7x^{\frac{6}{7}}}$

i $\frac{d}{dx}\left(5x^{\frac{2}{3}}\right) = \frac{10}{3}x^{-\frac{1}{3}} = \frac{10}{3x^{\frac{1}{3}}}$

j $\frac{d}{dx}\left(2x^{-\frac{1}{2}}\right) = -1x^{-\frac{3}{2}} = \frac{-1}{x^{\frac{3}{2}}}$

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5 **a** $\frac{d}{dx}\left(\frac{1}{x^5}\right)=\frac{d}{dx}x^{-5}=-5x^{-6}=\frac{-5}{x^6}$

b $\frac{d}{dx}\left(\frac{1}{x^6}\right)=\frac{d}{dx}x^{-6}=-6x^{-7}=\frac{-6}{x^7}$

c $\frac{d}{dx}\left(\frac{2}{x^3}\right)=\frac{d}{dx}2x^{-3}=-6x^{-4}=\frac{-6}{x^4}$

d $\frac{d}{dx}\left(\frac{1}{3x^2}\right)=\frac{d}{dx}\left(\frac{x^{-2}}{3}\right)=\frac{-2x^{-3}}{3}=\frac{-2}{3x^3}$

e $\frac{d}{dx}\left(\frac{4}{5x}\right)=\frac{d}{dx}\left(\frac{4x^{-1}}{5}\right)=\frac{-4x^{-2}}{5}=\frac{-4}{5x^2}$

f $\frac{d}{dx}\left(\sqrt{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{2}}\right)=\frac{1}{2}x^{-\frac{1}{2}}=\frac{1}{2\sqrt{x}}$

g $\frac{d}{dx}\left(\sqrt[6]{x}\right)=\frac{d}{dx}\left(x^{\frac{1}{6}}\right)=\frac{1}{6}x^{-\frac{5}{6}}=\frac{1}{6\sqrt[6]{x^5}}$

h $\frac{d}{dx}\left(4\sqrt[3]{x}\right)=\frac{d}{dx}\left(4x^{\frac{1}{3}}\right)=\frac{4}{3}x^{-\frac{2}{3}}=\frac{4}{3\sqrt[3]{x^2}}$

i $\frac{d}{dx}\left(\sqrt[3]{x^2}\right)=\frac{d}{dx}\left(x^{\frac{2}{3}}\right)=\frac{2}{3}x^{-\frac{1}{3}}=\frac{2}{3\sqrt[3]{x}}$

j $\frac{d}{dx}\left(\sqrt{x^5}\right)=\frac{d}{dx}\left(x^{\frac{5}{2}}\right)=\frac{5}{2}x^{\frac{3}{2}}=\frac{5\sqrt{x^3}}{2}$

6 **a** $f(x) = \frac{1}{x}$ so $f'(x) = -1x^{-2} = \frac{-1}{x^2}$ $\therefore f'(4) = \frac{-1}{16}$

b $g(x) = \frac{x^2-5}{x+3}$ so $g'(x) = \frac{2x\left(x+3\right)-1(x^2-5)}{(x+3)^2} = \frac{x^2+6x+5}{(x+3)^2}$

$\therefore g'\left(x\right)(-2) = \frac{4-12+5}{(-2+3)^2} = -3$

c $y = (2x^2 + 3x - 5)(x^3 - x^2 + 8)$

$\frac{dy}{dx} = \left(4x+3\right)\left(x^3 - x^2 + 8\right) + \left(3x^2 - 2x\right)\left(2x^2 + 3x - 5\right)$

$\therefore \left.\frac{dy}{dx}\right|_{x=3} = 15 \times 26 + 21 \times 22 = 852$

d $p(t) = 4t^{\frac{5}{3}} - t^{-\frac{4}{3}}$

$\therefore p'(t) = \frac{20}{3}t^{\frac{2}{3}} + \frac{4}{3}t^{-\frac{7}{3}}$

$p'(8) = \frac{20}{3}\times 4 + \frac{4}{3}\times 2^{-7} = \frac{80}{3} + \frac{1}{3\times 32} = 26\,\frac{65}{96} = 26.6770...$

e $h(y) = y^{-5}$

$h'(y) = -5y^{-6}$

$h'(\frac{1}{2}) = -5(2^6) = -320$

7 **a** $y = x^2(3x + 2)$

$\frac{dy}{dx} = 2x\left(3x+2\right) + 3x^2$

At $x = 4$, $\frac{dy}{dx} = 8\times 14 + 3\times 16 = 160$

b $y = \frac{1}{x}$

$\frac{dy}{dx} = \frac{-1}{x^2}$

At $x = 3$, $\frac{dy}{dx} = \frac{-1}{9}$

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c $y = \dfrac{2x+1}{x-2}$

$\dfrac{dy}{dx} = \dfrac{-5}{(x-2)^2}$

At $(1,-3)$, $\dfrac{dy}{dx} = -5$

Reasoning and communication

8 **a** $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right) = \dfrac{-2}{x^3}$

The derivative is positive for $x < 0$.

b $\dfrac{d}{dx}\left(x - \sqrt[3]{x}\right) = \dfrac{d}{dx}\left(x - x^{\frac{1}{3}}\right) = 1 - \frac{1}{3}x^{-\frac{2}{3}} = 1 - \dfrac{1}{3x^{\frac{2}{3}}} = 1 - \dfrac{1}{3\sqrt[3]{x^2}}$

$1 - \dfrac{1}{3\sqrt[3]{x^2}} > 0$

$\sqrt[3]{x^2} > \frac{1}{3}$

$x^2 > \frac{1}{27}$

$x < -\frac{1}{\sqrt{27}}$ or $x > \frac{1}{\sqrt{27}}$

c $\dfrac{d}{dx}\left(-6x^{-4}\right) = 24x^{-5} = \dfrac{24}{x^5}$

The derivative is positive for $x > 0$.

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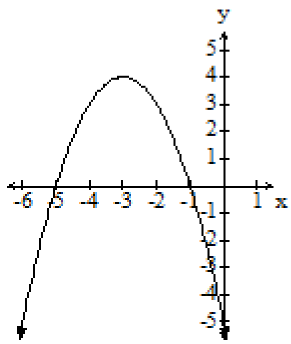
d $\frac{d}{dx}\left(\frac{x+3}{x^2-5}\right)=\frac{-x^2-6x-5}{\left(x^2-5\right)^2}=\frac{-(x+5)(x+1)}{\left(x^2-5\right)^2}$

$\left(x^2-5\right)^2$ is always positive

Consider the graph of $y = -(x + 5)(x + 1)$



$y > 0$ for $-5 < x < -1$

$\therefore$ The derivative is positive for $-5 < x < -1$.

e $\frac{d}{dx}(x^{-3}) = -3x^{-4} = -\frac{3}{x^4}$

$x^4 > 0$ for all values of x. $x \neq 0$

$-\frac{3}{x^4}$ is always negative (or not defined at $x = 0$)

The derivative is never positive.

9 $\frac{dy}{dx} = 2x(3x-2) + 3(x^2)$

$\frac{dy}{dx} = 9x^2 - 4x$

$\frac{dy}{dx} = 5$ then $5 = 9x^2 - 4x$

$9x^2 - 4x - 5 = 0$

$(x-1)(9x+5) = 0$

$x = 1$ or $x = -\frac{5}{9}$

If $x = 1$ then $y = 1$

If $x = -\frac{5}{9}$ then $y = \frac{25}{81}\left(-\frac{5}{3} - 2\right) = -\frac{275}{243}$

$M(1, 1), N\left(-\frac{5}{9}, 1\frac{32}{243}\right)$

10 $y = \frac{2x-1}{x+3}$

$\frac{dy}{dx} = \left(\frac{2(x+3) - 1(2x-1)}{(x+3)^2}\right)$

$\frac{dy}{dx} = \frac{7}{(x+3)^2}$

$\frac{7}{25} = \frac{7}{(x+3)^2}$

$(x+3)^2 = 25$

$x + 3 = \pm 5$

$x = -3 \pm 5$

$x = 2$ or $x = -8$

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11 $y = \frac{x^2 - 1}{x + 3}$

$\frac{dy}{dx} = \frac{2x(x+3) - 1(x^2 - 1)}{(x+3)^2}$

$\frac{dy}{dx} = \frac{x^2 + 6x + 1}{(x+3)^2}$

If $x = 2$, $\frac{dy}{dx} = \frac{4 + 12 + 1}{25} = \frac{17}{25}$

If $x = 2$, $y = \frac{3}{5}$

$y = mx + c$

$\frac{3}{5} = \frac{17}{25}(2) + c$

$c = -\frac{19}{25}$

Equation of tangent is $y = \frac{17}{25}x - \frac{19}{25}$

i.e. $25y - 17x + 19 = 0$

12 $C(x) = (x + 1)(0.04x^2 - 10x + 20)^2$

a $C(20) = 21(0.04 \times 20^2 - 200 + 20)^2$

$C(20) = \$564\,816$

b **i** $C'(x) = 1(0.04x^2 - 10x + 20)^2 + \left[\frac{d}{dx}(0.04x^2 - 10x + 20)^2\right] \times (x + 1)$

$C'(x) = (0.04x^2 - 10x + 20)^2 + 2(0.04x^2 - 10x + 20)(0.08x - 10) \times (x + 1)$

$C'(20) = \$84\,755.20$

ii $C'(50) = \$376\,960$

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13 **a** Show that $\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)\left(x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}+y^{\frac{3}{4}}\right)=x-y$.

$$\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)\left(x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}+y^{\frac{3}{4}}\right)$$
$$=x^{\frac{1}{4}}\left(x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}+y^{\frac{3}{4}}\right)-y^{\frac{1}{4}}\left(x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}+y^{\frac{3}{4}}\right)$$
$$=x+\cancel{x^{\frac{3}{4}}y^{\frac{1}{4}}}+\cancel{x^{\frac{1}{2}}y^{\frac{1}{2}}}+\cancel{x^{\frac{1}{4}}y^{\frac{3}{4}}}-\cancel{x^{\frac{3}{4}}y^{\frac{1}{4}}}-\cancel{x^{\frac{1}{2}}y^{\frac{1}{2}}}-\cancel{x^{\frac{1}{4}}y^{\frac{3}{4}}}-y$$
$$=x-y$$

b Show that $\frac{d}{dx}x^{\frac{1}{4}}=\frac{1}{4}x^{-\frac{3}{4}}$.

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f(x)=x^{\frac{1}{4}}$ and $f(x+h)=(x+h)^{\frac{1}{4}}$

so $f(x+h)-f(x)=(x+h)^{\frac{1}{4}}-x^{\frac{1}{4}}$

Using $\left(x^{\frac{1}{4}}-y^{\frac{1}{4}}\right)=\dfrac{x-y}{\left(x^{\frac{3}{4}}+x^{\frac{1}{2}}y^{\frac{1}{4}}+x^{\frac{1}{4}}y^{\frac{1}{2}}+y^{\frac{3}{4}}\right)}$

$$f(x+h)-f(x)=\frac{(x+h)-x}{\left[(x+h)^{\frac{3}{4}}+(x+h)^{\frac{1}{2}}x^{\frac{1}{4}}+(x+h)^{\frac{1}{4}}x^{\frac{1}{2}}+x^{\frac{3}{4}}\right]}$$
$$=\frac{h}{\left[(x+h)^{\frac{3}{4}}+(x+h)^{\frac{1}{2}}x^{\frac{1}{4}}+(x+h)^{\frac{1}{4}}x^{\frac{1}{2}}+x^{\frac{3}{4}}\right]}$$

$$\frac{f(x+h)-f(x)}{h}=\frac{1}{\left[(x+h)^{\frac{3}{4}}+(x+h)^{\frac{1}{2}}x^{\frac{1}{4}}+(x+h)^{\frac{1}{4}}x^{\frac{1}{2}}+x^{\frac{3}{4}}\right]}$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{1}{\left[(x+h)^{\frac{3}{4}}+(x+h)^{\frac{1}{2}}x^{\frac{1}{4}}+(x+h)^{\frac{1}{4}}x^{\frac{1}{2}}+x^{\frac{3}{4}}\right]}$$
$$=\frac{1}{\left(x^{\frac{3}{4}}+x^{\frac{3}{4}}+x^{\frac{3}{4}}+x^{\frac{3}{4}}\right)}=\frac{1}{4x^{\frac{3}{4}}}$$

$\therefore\ f'(x)=\frac{1}{4}x^{-\frac{3}{4}}$

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14 $(x-y)(x^2+xy+y^2) = x^3 + x^2y + xy^2 - x^2y - xy^2 - y^3$

$= x^3 - y^3$

Show that the derivative of $\sqrt[3]{x}$ is $\dfrac{1}{3\sqrt[3]{x^2}}$.

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$f(x) = \sqrt[3]{x}$ and $f(x+h) = \sqrt[3]{x+h}$

so $f(x+h)-f(x) = \sqrt[3]{x+h} - \sqrt[3]{x}$

Using $(x-y) = \dfrac{x^3-y^3}{\left(x^2+xy+y^2\right)}$

$$f(x+h)-f(x) = \frac{\left(\sqrt[3]{x+h}\right)^3 - \left(\sqrt[3]{x}\right)^3}{\left[\left(\sqrt[3]{x+h}\right)^2 + \left(\sqrt[3]{x+h}\right)\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right]}$$

$$= \frac{x+h-x}{\left[\left(\sqrt[3]{x+h}\right)^2 + \left(\sqrt[3]{x+h}\right)\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right]}$$

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\frac{\cancel{x}+h-\cancel{x}}{\left[\left(\sqrt[3]{x+h}\right)^2 + \left(\sqrt[3]{x+h}\right)\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right]}$$

$$= \left[\left(\sqrt[3]{x+h}\right)^2 + \left(\sqrt[3]{x+h}\right)\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right]$$

$$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{1}{\left[\left(\sqrt[3]{x+h}\right)^2 + \left(\sqrt[3]{x+h}\right)\sqrt[3]{x} + \left(\sqrt[3]{x}\right)^2\right]}$$

$$= \frac{1}{\left[\left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)^2 + \left(\sqrt[3]{x}\right)^2\right]} = \frac{1}{3\left(\sqrt[3]{x}\right)^2}$$

$$\therefore\ f'(x) = \frac{1}{3\sqrt[3]{x^2}}$$

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Exercise 1.02 The chain rule

Concepts and techniques

1 $m(x) = 3x + 3$, $g(x) = x^2$, $p(x) = \sqrt[3]{x}$ and $q(x) = x^2 + 4$

a $g(m) = m^2$

$m(x) = 3x + 3$

$g[m(x)] = g(3m + 3) = (3m + 3)^2 = 9m^2 + 18m + 9$

b $m(g) = 3g + 3$

$g(x) = x^2$

$m[g(x)] = m(3) = 3x^2 + 3$

c $q[m(x)] = q(3x + 3) = (3x + 3)^2 + 4 = 9x^2 + 18x + 13$

d $m[q(x)] = m(x^2 + 4) = 3(x^2 + 4) + 3 = 3x^2 + 15$

e $p[q(x)] = p(x^2 + 4) = \sqrt[3]{x^2+4}$

f $q[p(x)] = q(\sqrt[3]{x}) = \left(\sqrt[3]{x}\right)^2 + 4$

g $m[p(x)] = m(\sqrt[3]{x}) = 3\sqrt[3]{x} + 3$

$m[p(8)] = 3\sqrt[3]{8} + 3 = 9$

h $p[m(x)] = p(3x + 3) = \sqrt[3]{3x+3}$

$p[m(8)] = \sqrt[3]{3\times8+3} = \sqrt[3]{27} = 3$

i $g[q(x)] = g(x^2 + 4) = (x^2 + 4)^2$

$g[q(3)] = (9 + 4)^2 = 169$

j $q[g(x)] = q(x^2) = x^4 + 4$

$q[g(3)] = 81 + 4 = 85$

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2 **a** $\frac{d}{dx}(x+3)^4 = 4(x+3)^3 \times 1 = 4(x+3)^3$

b $\frac{d}{dx}(2x-1)^3 = 3(2x-1)^2 \times 2 = 6(2x-1)^2$

c $\frac{d}{dx}(5x^2-4)^7 = 7(5x^2-4)^6 \times 10x = 70x(5x^2-4)^6$

d $\frac{d}{dx}(8x+3)^6 = 6(8x+3)^5 \times 8 = 48(8x+3)^5$

e $\frac{d}{dx}(1-x)^5 = 5(1-x)^4\left(-1\right) = -5(1-x)^4$

3 **a** $\frac{d}{dx}3(5x+9)^9 = 27(5x+9)^8 \times 5 = 135(5x+9)^8$

b $\frac{d}{dx}2(x-4)^2 = 4(x-4) \times 1 = 4x-16$

c $\frac{d}{dx}(2x^3+3x)^4 = 4(2x^3+3x)^3 \times \left(6x^2+3\right)$

d $\frac{d}{dx}(x^2+5x-1)^8 = 8(x^2+5x-1)^7\left(2x+5\right)$

e $\frac{d}{dx}(x^6-2x^2+3)^6 = 12(x^6-2x^2+3)^5\left(3x^5-2x\right)$

4 **a** $\frac{d}{dx}(3x-1)^{\frac{1}{2}} = \frac{1}{2}(3x-1)^{-\frac{1}{2}} \times 3 = \frac{3}{2\sqrt{3x-1}}$

b $\frac{d}{dx}(4-x)^{-2} = -2(4-x)^{-3} \times \left(-1\right) = 2(4-x)^{-3}$

c $\frac{d}{dx}(x^2-9)^{-3} = -3(x^2-9)^{-4} \times \left(2x\right) = -6x(x^2-9)^{-4}$

d $\frac{d}{dx}(5x+4)^{\frac{1}{3}} = \frac{1}{3}(5x+4)^{-\frac{2}{3}} \times 5 = \frac{5}{3(5x+4)^{\frac{2}{3}}}$

e $\frac{d}{dx}(x^3-7x^2+x)^{\frac{3}{4}} = \frac{3}{4}(x^3-7x^2+x)^{-\frac{1}{4}} \times (3x^2-14x+1) = \frac{3(3x^2-14x+1)}{4(x^3-7x^2+x)^{\frac{1}{4}}}$

5 **a** $\frac{d}{dx}\sqrt{3x+4}=\frac{d}{dx}\left(3x+4\right)^{\frac{1}{2}}=\frac{1}{2}\left(3x+4\right)^{-\frac{1}{2}}\times 3=\frac{3}{2\sqrt{\left(3x+4\right)}}$

b $\frac{d}{dx}\frac{1}{5x-2}=\frac{d}{dx}\left(5x-2\right)^{-1}=-1\left(5x-2\right)^{-2}\times 5=\frac{-5}{\left(5x-2\right)^{2}}$

c $\frac{d}{dx}\frac{1}{(x^2+1)^4}=\frac{d}{dx}(x^2+1)^{-4}=-4(x^2+1)^{-5}\times 2x=\frac{-8x}{(x^2+1)^5}$

d $\frac{d}{dx}\sqrt[3]{(7-3x)^2}=\frac{d}{dx}(7-3x)^{\frac{2}{3}}=\frac{2}{3}(7-3x)^{-\frac{1}{3}}\times\left(-3\right)=\frac{-2}{(7-3x)^{\frac{1}{3}}}=\frac{-2}{\sqrt[3]{7-3x}}$

e $\frac{d}{dx}\frac{5}{\sqrt{4+x}}=\frac{d}{dx}5\left(4+x\right)^{-\frac{1}{2}}=\frac{-5}{2}\left(4+x\right)^{-\frac{3}{2}}\times 1=\frac{-5}{2(4+x)^{\frac{3}{2}}}=\frac{-5}{2\sqrt{\left(4+x\right)^{3}}}$

6 **a** $\frac{d}{dx}x^2(x+1)^3$

$=2x(x+1)^3+3(x+1)^2\times 1\times x^2$

$=x(x+1)^2\left[2(x+1)+3x\right]$

$=x(x+1)^2\left(5x+2\right)$

b $\frac{d}{dx}4x(3x-2)^5$

$=4(3x-2)^5+5(3x-2)^4\times 3\times 4x$

$=4(3x-2)^4\left(3x-2+15x\right)$

$=4(3x-2)^4\left(18x-2\right)$

$=8(3x-2)^4\left(9x-1\right)$

c $\frac{d}{dx}3x^4(4-x)^3$

$=12x^3(4-x)^3+3(4-x)^2\left(-1\right)3x^4$

$=3x^3(4-x)^2\left[4\left(4-x\right)-3x\right]$

$=3x^3(4-x)^2\left(16-7x\right)$

d $\frac{d}{dx}(x+1)(2x+5)^4$

$=1(2x+5)^4+4(2x+5)^3\times 2\times(x+1)$

$=(2x+5)^3\left(2x+5+8x+8\right)$

$=(2x+5)^3\left(10x+13\right)$

e $\frac{d}{dx}(x^3+5x^2-3)(x^2+1)^5$

$=(3x^2+10x)(x^2+1)^5+5(x^2+1)^4\times 2x\times(x^3+5x^2-3)$

$=x(x^2+1)^4\left[\left(3x+10\right)(x^2+1)+10(x^3+5x^2-3)\right]$

$=x(x^2+1)^4\left[\left(3x^3+10x^2+3x+10\right)+(10x^3+50x^2-30)\right]$

$=x(x^2+1)^4\left(13x^3+60x^2+3x-20\right)$

7 **a** $\frac{d}{dx}\frac{(2x-9)^3}{5x+1}$

$=\frac{(2x-9)^3}{5x+1}$

$=\frac{3(2x-9)^2 2\left(5x+1\right)-5(2x-9)^3}{\left(5x+1\right)^2}$

$=\frac{(2x-9)^2\left(6\left(5x+1\right)-5(2x-9)\right)}{\left(5x+1\right)^2}$

$=\frac{(2x-9)^2\left(30x+6-10x+45\right)}{\left(5x+1\right)^2}$

$=\frac{\left(20x+51\right)(2x-9)^2}{\left(5x+1\right)^2}$

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b $\frac{d}{dx}\frac{x-1}{(7x+2)^4}$

$$=\frac{1(7x+2)^4-4(7x+2)^3 7(x-1)}{(7x+2)^8}$$

$$=\frac{(7x+2)^3\left((7x+2)-28(x-1)\right)}{(7x+2)^8}$$

$$=\frac{(7x+2)^3\left(-21x+30\right)}{(7x+2)^8}$$

$$=\frac{3\left(10-7x\right)}{(7x+2)^5}$$

c $\frac{d}{dx}\frac{(3x+4)^5}{(2x-5)^3}$

$$=\frac{5(3x+4)^4 3(2x-5)^3-3(2x-5)^2 2(3x+4)^5}{(2x-5)^6}$$

$$=\frac{3(3x+4)^4(2x-5)^2\left[5(2x-5)^1-2(3x+4)\right]}{(2x-5)^6}$$

$$=\frac{3(3x+4)^4\left(4x-33\right)}{(2x-5)^4}$$

d $\frac{d}{dx}\frac{3x+1}{\sqrt{x+1}}$

$$=\frac{3\sqrt{x+1}-\frac{1}{2}(x+1)^{-\frac{1}{2}}\times 1\left(3x+1\right)}{\left(\sqrt{x+1}\right)^2}$$

$$=\frac{1}{(x+1)}\times\left(3\sqrt{x+1}-\frac{3x+1}{2\sqrt{x+1}}\right)$$

$$=\frac{1}{(x+1)}\times\left[\frac{6\left(x+1\right)-\left(3x+1\right)}{2\sqrt{x+1}}\right]$$

$$=\left[\frac{3x+5}{2\sqrt{\left(x+1\right)^3}}\right]$$

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e $\frac{d}{dx}\frac{\sqrt{x-1}}{2x-3}$

$$=\frac{\frac{1}{2}\left(x-1\right)^{-\frac{1}{2}}\times 1\times\left(2x-3\right)-2\sqrt{x-1}}{\left(2x-3\right)^2}$$

$$=\frac{1}{\left(2x-3\right)^2}\times\left(\frac{2x-3}{2\sqrt{x-1}}-2\sqrt{x-1}\right)$$

$$=\frac{1}{\left(2x-3\right)^2}\times\left[\frac{2x-3-4(x-1)}{2\sqrt{x-1}}\right]$$

$$=\left[\frac{-2x+1}{2\left(2x-3\right)^2\sqrt{x-1}}\right]$$

8 **a** $g(x) = (3x + 5)^3(2x - 1)^2$

$g'(x) = 3(3x + 5)^2 3(2x - 1)^2 + 2(2x - 1)2(3x + 5)^3$

$= (3x + 5)^2(2x - 1)[9(2x - 1) + 4(3x + 5)]$

$g'(x) = (3x + 5)^2(2x - 1)[30x + 11]$

$g'(-2) = (-1)^2(-5)[-49] = 245$

b $f(t) = \frac{3t^2-2t+1}{(4t-5)^3}$

$$f'\left(t\right)=\frac{(6t-2)(4t-5)^3-3(4t-5)^2 4\left(3t^2-2t+1\right)}{(4t-5)^6}$$

$$=\frac{(4t-5)^2\left[(6t-2)(4t-5)-12\left(3t^2-2t+1\right)\right]}{(4t-5)^6}$$

$$f'\left(3\right)=\frac{\left[16\times 7-12(22)\right]}{(7)^4}=\frac{-152}{2401}$$

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c $p(x)=\sqrt[3]{(4x-7)^5}=(4x-7)^{\frac{5}{3}}$

$p'(x)=\frac{5}{3}(4x-7)^{\frac{2}{3}}\times 4$

$p'\left(0.5\right)=\frac{5}{3}(2-7)^{\frac{2}{3}}\times 4=19.49$

d $h(z)=(3z-1)^4\left(\sqrt[3]{2z-3}\right)^2$

$h'(z)=4(3z-1)^3\times 3\left(\sqrt[3]{2z-3}\right)^2+\frac{2}{3}(2z-3)^{-\frac{1}{3}}\times 2(3z-1)^4$

$h'(z)=12(3z-1)^3\left(\sqrt[3]{2z-3}\right)^2+\frac{4(3z-1)^4}{3\sqrt[3]{2z-3}}$

$h'(-1)=12(-4)^3\left(\sqrt[3]{-5}\right)^2+\frac{4(-4)^4}{3\sqrt[3]{-5}}=-2445.26$

e $m(x)=\frac{1}{(3x^2-2x-4)^3}=(3x^2-2x-4)^{-3}$

$m'\left(x\right)=-3(3x^2-2x-4)^{-4}\left(6x-2\right)$

$m'(1)=-3(-3)^{-4}4=\frac{-4}{27}$

Reasoning and communication

9 $y = (2x - 3)^4$

$$\frac{dy}{dx} = 4\left(2x-3\right)^3 \times 2 = 8\left(2x-3\right)^3$$

$$-8 = 8\left(2x-3\right)^3$$

$$\left(2x-3\right)^3 = -1$$

$$2x-3 = -1$$

$$2x = 2$$

$$x = 1 \quad y = ? \quad y = 1$$

Point is $\left(1, 1\right)$

10 $V = (1500t + 17t^2)^3$

a $V_{30} = 2.2 \times 10^{14}$ litres

b $$\frac{dV}{dt} = 3\left(1500t + 17t^2\right)^2\left(1500 + 34t\right)$$

At $t = 5$,

$$\frac{dV}{dt} = 3(1500 \times 5 + 17 \times 25)^2\left(1500 + 34 \times 5\right) = 3.14 \times 10^{11} \text{ litres per minute}$$

c At $t = 30$,

$$\frac{dV}{dt} = 3(1500 \times 30 + 17 \times 900)^2\left(1500 + 34 \times 30\right) = 2.75 \times 10^{13} \text{ litres per minute}$$

11 Given $q(x) = \sqrt{x-4}$

a $x - 4 \geq 0$ so $x \geq 4$

b $q'(x) = \frac{1}{2}(x-4)^{-\frac{1}{2}} \times 1 = \frac{1}{2\sqrt{x-4}}$

At $x = 13$ the gradient is $q'(13) = \frac{1}{6}$

c $q(13) = \sqrt{13-4} = 3$

$y = \frac{x}{6} + c$

At (13, 3), $3 = \frac{13}{6} + c \Rightarrow c = \frac{5}{6}$

The equation of the tangent at $x = 13$ is $y = \frac{x}{6} + \frac{5}{6}$

d At $x = a$, $q'(a) = \frac{1}{2\sqrt{a-4}}$

e $A(a, \sqrt{a-4})$, Gradient is $\frac{\sqrt{a-4} - 0}{a - 0} = \frac{\sqrt{a-4}}{a}$

f $y = \frac{1}{2\sqrt{a-4}}x + c$

Goes through (0, 0) $\Rightarrow c = 0$

Using A,

$\sqrt{a-4} = \frac{a}{2\sqrt{a-4}}$

$2(a - 4) = a$

$a = 8$

$q(8) = \sqrt{8-4} = 2$

$A(8, 2)$

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12 For the function $y = \frac{1}{x+2} + 2$

a $\frac{dy}{dx} = -1(x+2)^{-2} = \frac{-1}{(x+2)^2}$

At $x = -2.5$

$\frac{dy}{dx} = \frac{-1}{(-2.5+2)^2} = -4$

b At $x = -1.5$

$\frac{dy}{dx} = \frac{-1}{(-1.5+2)^2} = -4$

c They are parallel.

d At $x = -2.2$, $\frac{dy}{dx} = -25$

e At $x = -1.8$, $\frac{dy}{dx} = -25$

f They are parallel.

g At $x = -3$, $\frac{dy}{dx} = -1$

h At $x = -1$, $\frac{dy}{dx} = -1$

i They are parallel.

j $x = -2 \pm k$

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13 $y = (x + 2)^3$, so $\frac{dy}{dx} = 3(x + 2)^2$

At $x = -3$, $\frac{dy}{dx} = 3$

The gradient of the normal is $-\frac{1}{3}$

Equation of the normal: $y = -\frac{x}{3} + c$

Substitute the point $(-3, -1) \Rightarrow -1 = -\frac{-3}{3} + c \Rightarrow c = -2$

Therefore the normal passes through the point (0, –2) so has *y*-intercept –2.

14 $y = 4x^2$, Points equidistant from the origin are $(a, 4a^2)$ and $(-a, 4a^2)$

$$\frac{dy}{dx} = 8x$$

If $x = a$, $\frac{dy}{dx} = 8a$ and if $x = -a$, $\frac{dy}{dx} = -8a$

Slopes of the normals are

$m_\perp = -\frac{1}{8a}$ and $m_\perp = \frac{1}{8a}$

Equations of the normals are

$y = -\frac{x}{8a} + c$ and $y = \frac{x}{8a} + c$

Use the points $(a, 4a^2)$ and $(-a, 4a^2)$ to find the c values:

$4a^2 = -\frac{a}{8a} + c$ and $4a^2 = \frac{-a}{8a} + c$

$c = 4a^2 + \frac{1}{8}$ $\qquad$ $c = 4a^2 + \frac{1}{8}$

Equations of the normals are

$y = -\frac{x}{8a} + 4a^2 + \frac{1}{8}$ and $y = \frac{x}{8a} + 4a^2 + \frac{1}{8}$

On the x-axis, $y = 0$, so

$\frac{x}{8a} = 4a^2 + \frac{1}{8}$ and $\frac{x}{8a} = -4a^2 - \frac{1}{8}$

$x = 32a^3 + a$ and $x = -32a^3 - a$

These are equidistant distances from the origin on the x-axis.

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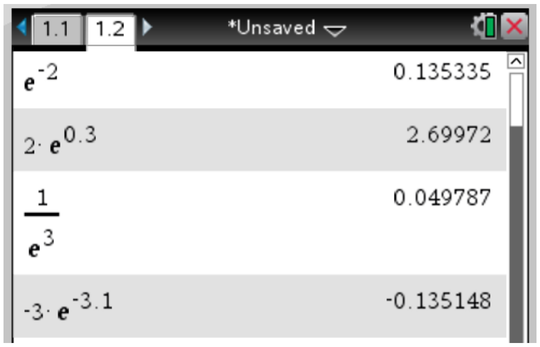
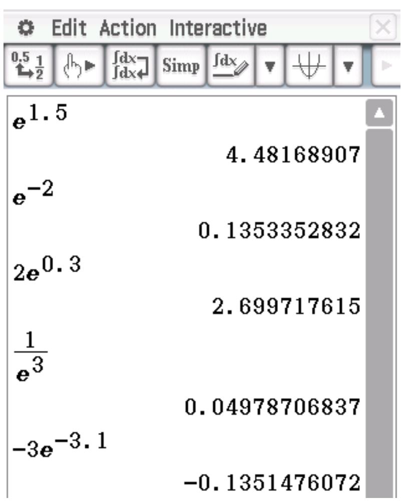
Exercise 1.03 The derivative of exponential functions

Concepts and techniques

1 TI-Nspire CAS



ClassPad



a $e^{1.5} = 4.48$

b $e^{-2} = 0.14$

c $2e^{0.3} = 2.70$

d $\dfrac{1}{e^3} = 0.05$

e $-3e^{-3.1} = -0.14$

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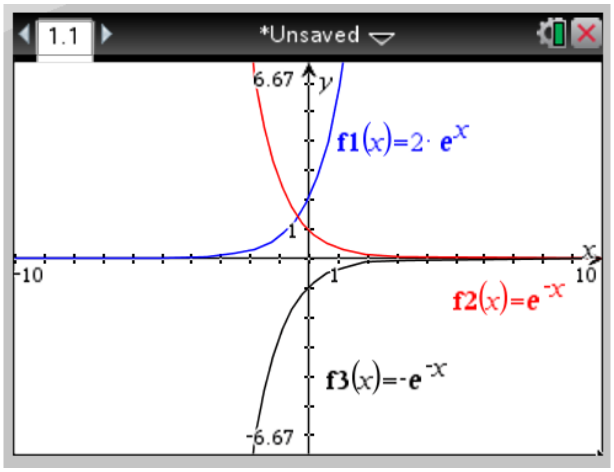
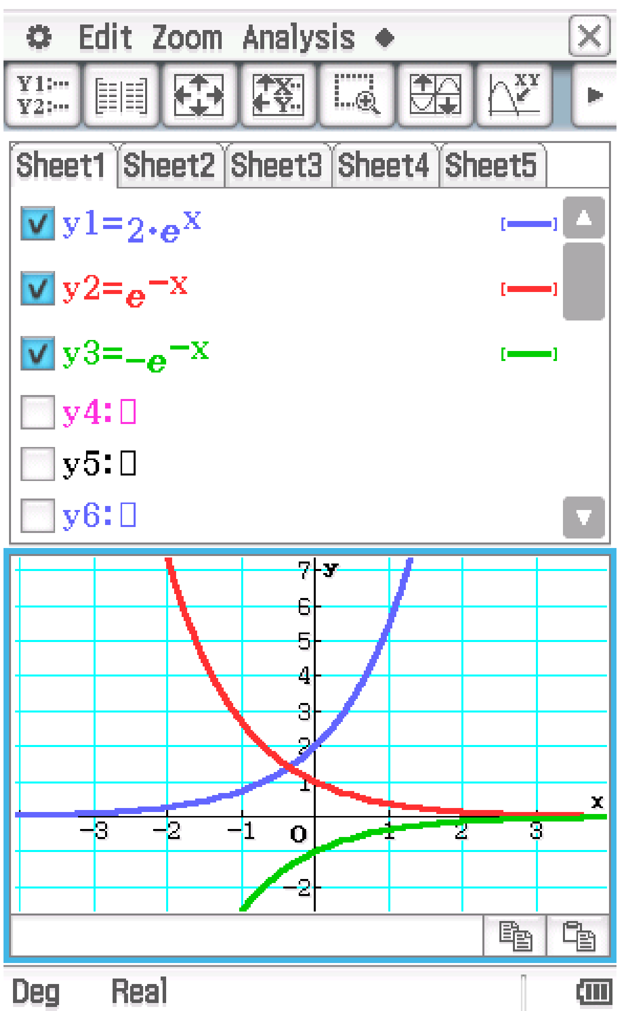
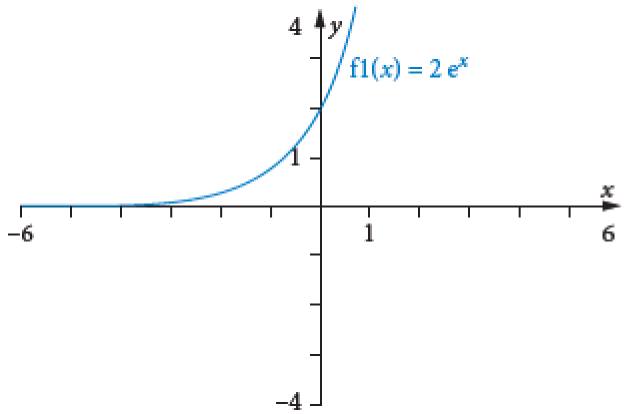
2 TI-Nspire CAS



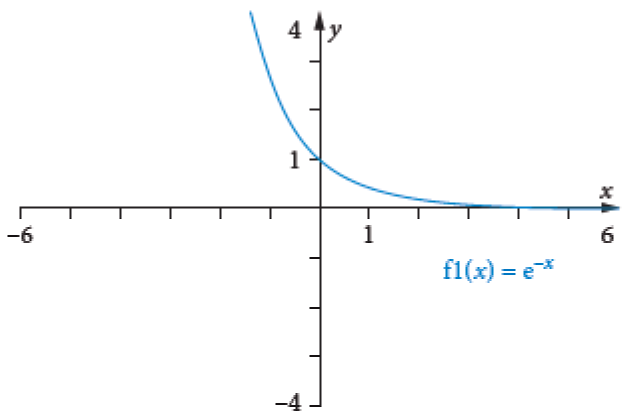
ClassPad



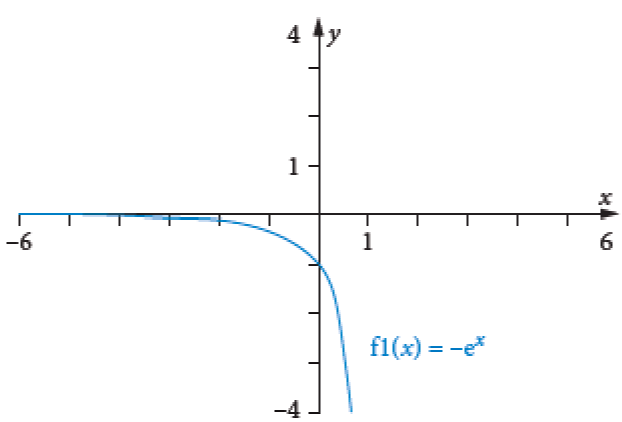
a $y = 2e^{x}$



b $y = e^{-x}$



c $y = -e^{x}$



3 **a** $y = e^x$, so $\frac{dy}{dx} = e^x$

At $(1, e)$, $\frac{dy}{dx} = e$

b $y = e^x$, so $\frac{dy}{dx} = e^x$

At $x = 0.58$, $\frac{dy}{dx} = e^{0.58} = 1.79$ (2 dp)

c $y = e^x$, so $\frac{dy}{dx} = e^x$

At $\left(-1, \frac{1}{e}\right)$, $\frac{dy}{dx} = e^{-1}$

Equation of tangent is $y = e^{-1}x + c$

Substitute $\left(-1, \frac{1}{e}\right)$ to find c

$$\frac{1}{e} = \frac{1}{e}(-1) + c$$

$$c = \frac{2}{e}$$

$\therefore$ The equation of the tangent is $y = \frac{x}{e} + \frac{2}{e}$

4 **a** $\frac{d}{dx} 9e^x = 9e^x$

b $\frac{d}{dx}(-e^x) = -e^x$

c $\frac{d}{dx}(e^x + x^2) = e^x + 2x$

d $\frac{d}{dx}(2x^3 - 3x^2 + 5x - e^x) = 6x^2 - 6x + 5 - e^x$

e $\frac{d}{dx}(e^x + 1)^3 = 3e^x(e^x + 1)^2$

f $\frac{d}{dx}(5 - e^x)^9 = 9(5 - e^x)^8(-e^x) = -9e^x(5 - e^x)^8$

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g $\frac{d}{dx}(2e^x - 3)^6 = 6(2e^x - 3)^5(2e^x) = 12e^x(2e^x - 3)^5$

h $\frac{d}{dx}\ (e^x + x)^4 = 4(e^x + x)^3(e^x + 1)$

5 **a** $\frac{d}{dx}e^{3x} = 3e^{3x}$

b $\frac{d}{dx}e^{2x-1} = 2e^{2x-1}$

c $\frac{d}{dx}2e^{4x} = 2\times4\times e^{4x} = 8e^{4x}$

d $\frac{d}{dx}\ e^{x^2-1} = 2xe^{x^2-1}$

e $\frac{d}{dx}\ e^{2x^5-3x^3+x-3} = \left(10x^4 - 9x^2 + 1\right)e^{2x^5-3x^3+x-3}$

6 **a** $\frac{d}{dx}xe^x = 1\times e^x + e^x x = e^x + xe^x = e^x(1 + x)$

b $\frac{d}{dx}(2x + 3)e^x = 2e^x + e^x(2x + 3) = (2x + 5)e^x$

c $\frac{d}{dx}\ 5x^3e^x = 15x^2e^x + e^x5x^3 = 5\left(3+x\right)x^2e^x$

d $\frac{d}{dx}\ 2xe^{3x} = 2e^{3x} + 3e^{3x}2x = 2\left(1+3x\right)e^{3x}$

e $\frac{d}{dx}\ e^{2x}\left(x^2 + x + 2\right) = 2e^{2x}\left(x^2 + x + 2\right) + \left(2x+1\right)e^{2x} = \left(2x^2 + 4x + 5\right)e^{2x}$

7 **a** $\frac{d}{dx}\ \frac{e^x}{x^2} = \frac{e^xx^2 - 2xe^x}{x^4} = \frac{\left(x^2 - 2x\right)e^x}{x^4} = \frac{\left(x - 2\right)e^x}{x^3}$

b $\frac{d}{dx}\ \frac{e^{6x}}{3x} = \frac{6e^{6x}3x - 3e^{6x}}{9x^2} = \frac{\left(18x - 3\right)e^{6x}}{9x^2} = \frac{\left(6x - 1\right)e^{6x}}{3x^2}$

c $\frac{d}{dx}\ \frac{2e^{5x}}{5x^3} = \frac{10e^{5x}\times5x^3 - 15x^2 2e^{5x}}{25x^6} = \frac{\left(50x^3 - 30x^2\right)e^{5x}}{25x^6} = \frac{2\left(5x - 3\right)e^{5x}}{5x^4}$

d $\frac{d}{dx}\ \frac{x-1}{e^x} = \frac{1\times e^x - e^x\left(x-1\right)}{e^{2x}} = \frac{\left(2-x\right)e^x}{e^{2x}} = \frac{\left(2-x\right)}{e^x}$

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e $\frac{d}{dx}\frac{e^x+1}{e^{2x}}=\frac{e^xe^{2x}-2e^{2x}\left(e^x+1\right)}{e^{4x}}=\frac{e^{2x}\left[e^x-2\left(e^x+1\right)\right]}{e^{4x}}=\frac{-\left(e^x+2\right)}{e^{2x}}$

8 **a** $g(x) = \frac{e^x-4}{\sqrt{e^x+1}}$

$$g'\left(x\right) = \frac{e^x\sqrt{e^x+1}-\frac{1}{2}\left(e^x+1\right)^{\frac{-1}{2}}e^x\left(e^x-4\right)}{e^x+1}$$

$$=\frac{e^x}{e^x+1}\left[\sqrt{e^x+1}-\frac{\left(e^x-4\right)}{2\sqrt{e^x+1}}\right]$$

$$=\frac{e^x}{e^x+1}\left[\frac{2\left(e^x+1\right)-\left(e^x-4\right)}{2\sqrt{e^x+1}}\right]$$

$$=\frac{e^x}{e^x+1}\left(\frac{e^x+6}{2\sqrt{e^x+1}}\right)$$

$$=\frac{e^x\left(e^x+6\right)}{2\sqrt{\left(e^x+1\right)^3}}$$

$$\therefore g'(3) = \ =\frac{e^3\left(e^3+6\right)}{2\sqrt{\left(e^3+1\right)^3}}$$

$g'(3) = 2.71$ (2 dp)

b $y = e^{4x}(x^3 - 3x + 5)$

$$\frac{dy}{dx} = 4e^{4x}(x^3 - 3x + 5) + (3x^2 - 3)e^{4x}$$

$$= e^{4x}(4x^3 + 3x^2 - 12x + 17)$$

$$\left.\frac{dy}{dx}\right|_{x=-1} = ?$$

$$\frac{dy}{dx} = e^{-4}\,(-4 + 3 + 12 + 17)$$

$$\therefore \frac{dy}{dx} = 28e^{-4}$$

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c $f(x) = \dfrac{xe^{3x}+5}{x^2+e}$

$f'(x) = \dfrac{\left(e^{3x}+3xe^{3x}\right)\left(x^2+e\right)-\left(2x\right)\left(xe^{3x}+5\right)}{\left(x^2+e\right)^2}$

$f'(2) = 348.4$ (1 dp)

d $h(x) = 5x^2e^{3x} + e^x$

$h'(x) = 10xe^{3x}+3\ e^{3x}5x^2 + e^x$

$h'(2) = 20e^6+3\ e^{6x}20 + e^2$

$h'(2) = 80e^6 + e^2 = 32\ 281.69$

e $y = 5e^{x^2-x-6}$

$\dfrac{dy}{dx} = 5(2x-1)\,e^{x^2-x-6}$

At $x = 1$, $\dfrac{dy}{dx} = 5\,e^{-6} = 0.012$

At $x = 3$, $\dfrac{dy}{dx} = 25\,e^{0} = 25$

Reasoning and communication

9 Let $y = xe^{2x-1}$

$\dfrac{dy}{dx} = 1\times e^{2x-1} + 2e^{2x-1}x = e^{2x-1}(1+2x)$

$x = ?$, for $\dfrac{dy}{dx} = 5e^3$

$5e^3 = e^{2x-1}(1+2x)$

We have $3 = 2x - 1$ (equating the power of e) and $1 + 2x = 5$ (equating the coefficients).

Both give $x = 2$.

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10 $p(x) = e^{-kx} + 3x$

$p(2) = e^{-2k} + 6 \Rightarrow (2, e^{-2k} + 6)$

$p'(x) = -ke^{-kx} + 3$

$p'(2) = -ke^{-2k} + 3 \Rightarrow$ gradient of the tangent is $-ke^{-2k} + 3$ at $x = 2$

Equation of the tangent:

$y = (-ke^{-2k} + 3)x + c$

Using $(2, e^{-2k} + 6)$

$e^{-2k} + 6 = (-ke^{-2k} + 3)2$ ($c = 0$, as passing through the origin)

$e^{-2k} + 6 = -2ke^{-2k} + 6$

$-2k = 1$ (equating coefficients)

$k = -\frac{1}{2}$

Exercise 1.04 Applications of the exponential function and its derivative

Reasoning and communication

1 $N = 175e^{0.062t}$

a **i** $t = 0$, so $N = 175e^{0} = 175$ cases

ii $t = 1$, so $N = 175e^{0.062} = 186$ cases

iii $t = 6$, so $N = 175e^{0.062 \times 6} = 254$ cases

iv $t = 26$, so $N = 175e^{0.062 \times 26} = 877$ cases

b $\dfrac{dN}{dt} = 175 \times 0.062e^{0.062t} = 10.85 \times e^{0.062t}$

i $t = 1$, $\dfrac{dN}{dt} = 11.54$ cases per week

ii $t = 6$, $\dfrac{dN}{dt} = 15.74$ cases per week

iii $t = 26$, $\dfrac{dN}{dt} = 54.39$ cases per week

2 $S = 180e^{0.12t}$

a $S_0 = 180e^{0} = 180$ sales

b $S_{14} = 180e^{0.12 \times 14} = 965.80\ldots$ sales

c $\dfrac{dS}{dt} = 180 \times 0.12e^{0.12t}$ sales per day

At $t = 14$, $\dfrac{dS}{dt} = 180 \times 0.12e^{0.12 \times 14} = 115.89...$ sales per day

d At $t = 42$, $\dfrac{dS}{dt} = 180 \times 0.12e^{0.12 \times 42} = 3336.55...$ sales per day

3 $N = 1100e^{0.025t}$

A study of swans in an area of Western Australia showed that the numbers were gradually increasing, with the number of swans N over t months given by $N = 1100e^{0.025t}$.

a $N_0 = 1100e^0 = 1100$

b **i** $N_5 = 1100e^{0.025 \times 5} = 1246$ swans

ii $N_{12} = 1100e^{0.025 \times 12} = 1485$ swans

iii $N_{36} = 1100e^{0.025 \times 36} = 2706$ swans

c $\frac{dN}{dt} = 1100 \times 0.025e^{0.025t}$

i At $t = 0$, $\frac{dN}{dt} = 1100 \times 0.025e^0 = 27.5$ i.e. 27 or 28

ii At $t = 5$, $\frac{dN}{dt} = 1100 \times 0.025e^{0.025 \times 5} = 31$

iii At $t = 12$, $\frac{dN}{dt} = 1100 \times 0.025e^{0.025 \times 12} = 37$

4 $M = 200e^{-0.012t}$

a $M_0 = 200e^0 = 200$ grams

b **i** $M_5 = 200e^{-0.012 \times 5} = 188.35$ grams

ii $M_{20} = 200e^{-0.012 \times 20} = 157.33$ grams

iii $M_{100} = 200e^{-0.012 \times 100} = 60.24$ grams

c $\frac{dM}{dt} = 200 \times (-0.012)e^{-0.012t}$

$\frac{dM}{dt} = -2.4e^{-0.012t}$

i At $t = 5$, $\frac{dM}{dt} = -2.4e^{-0.012 \times 5} = -2.26$, i.e. decaying 2.26 grams per year

ii At $t = 20$, $\frac{dM}{dt} = -2.4e^{-0.012 \times 20} = -1.89$, i.e. decaying 1.89 grams per year

iii At $t = 100$, $\frac{dM}{dt} = -2.4e^{-0.012 \times 100} = -0.72$, i.e. decaying 0.72 grams per year

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5 $A = 120\,000e^{-0.033t}$

a **i** $A_{10} = 120\,000e^{-0.033 \times 10} = 86\,271$ hectares

ii $A_{25} = 120\,000e^{-0.033 \times 25} = 52\,588$ hectares

iii $A_{50} = 120\,000e^{-0.033 \times 50} = 23\,046$ hectares

b $\frac{dM}{dt} = 120\,000 \times (-0.033)e^{-0.033t}$

$\frac{dM}{dt} = -3960e^{-0.033t}$

i At $t = 2$, $\frac{dM}{dt} = -3960e^{-0.033 \times 2} = -3707$

i.e. decreasing at 3707 hectares per year

ii At $t = 15$, $\frac{dM}{dt} = -3960e^{-0.033 \times 15} = -2414$ hectares per year

i.e. decreasing at 2414 hectares per year

iii At $t = 40$, $\frac{dM}{dt} = -3960e^{-0.033 \times 40} = -1058$ hectares per year

i.e. decreasing at 1058 hectares per year

6 $\frac{dN}{dt} = 0.29N \Rightarrow N = Ae^{0.29t}$

a $N_0 = 90\,000$

$\therefore N = 90\,000e^{0.29t}$

b $N_6 = 90\,000e^{0.29 \times 6} = 512\,761$ bacteria

c $\frac{dN}{dt} = 90\,000 \times 0.29e^{0.29t}$

$\frac{dN}{dt} = 26100e^{0.29t}$

At $t = 6$, $\frac{dN}{dt} = 26100e^{0.29 \times 6} = 148\,701$ bacteria per hour

d At $t = 10$, $\frac{dN}{dt} = 26\,100e^{0.29 \times 10} = 474\,345$ bacteria per hour

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7 $\frac{dR}{dt} = -0.008R \Rightarrow R = Ae^{-0.008t}$

a $R_0 = 43$ cm $\Rightarrow R = 43e^{-0.008t}$

b **i** $t = 10$, $R = 43e^{-0.008 \times 10} = 39.7$ cm

ii $t = 30$, $R = 43e^{-0.008 \times 30} = 33.8$ cm

iii $t = 100$, $R = 43e^{-0.008 \times 100} = 19.3$ cm

c $\frac{dR}{dt} = -0.008R$

i $t = 10$, $\frac{dR}{dt} = -0.32$ so decreasing at 0.32 cm per year

ii $t = 30$, $\frac{dR}{dt} = -0.27$ so decreasing at 0.27 cm per year

iii $t = 100$, $\frac{dR}{dt} = -0.15$ so decreasing at 0.15 cm per year

8 $P = P_0e^{0.024t}$

a P_0

b $P_6 = P_0e^{0.024 \times 6} = P_0(1.154\ 884)$

Increase is 15.5%

c $\frac{dP}{dt} = 0.024,\ P_0e^{0.024t} = 0.024P$

9 $Q = Q_0e^{-0.07t}$

The formula for the decay of a radioactive substance over t years is given by $Q = Q_0e^{-0.07t}$.

a **i** $t = 2$ years, $Q = Q_0(e^{-0.07 \times 2}) = Q_0(0.8693\ldots)$ so about 87% left

ii $t = 10$ years, $Q = Q_0(e^{-0.07 \times 10}) = Q_0(0.4965\ldots)$ so about 49.7% left

iii $t = 20$ years, $Q = Q_0(e^{-0.07 \times 20}) = Q_0(0.2465\ldots)$ so about 24.7% left

b $0.5Q_0 = Q_0e^{-0.07t}$, $t = ?$

$0.5 = e^{-0.07t}$

$t = 9.9021\ldots$ years i.e. the half-life is about 10 years

10 **a** $y = e^{2x}$

$\frac{dy}{dx} = 2e^{2x}$

At $M\ (2, e^4)$, $\frac{dy}{dx} = 2e^4$

Equation of the tangent at M:

$y = 2e^4x + c$

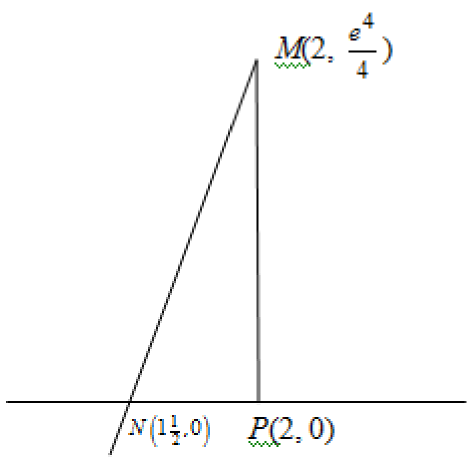
Using $M\ (2, e^4)$, $e^4 = 2e^4(2) + c \Rightarrow c = e^4 - 4e^4 = -3e^4$

$y = 2e^4x - 3e^4$

b If $y = 0$, then $x = \frac{3e^4}{2e^4} = 1\tfrac{1}{2}$

$N\left(1\tfrac{1}{2}, 0\right)$

c $A_\Delta = \frac{1}{2} \times \left(2 - 1\tfrac{1}{2}\right) \times e^4 = \frac{e^4}{4}$



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11 $P = 100 + 2e^{0.3t}$

a $P_0 = 102$

b **i** $P_6 = 100 + 2e^{0.3 \times 6} = 112$ waterbirds

ii $P_{24} = 100 + 2e^{0.3 \times 24} = 2779$ waterbirds

c $\frac{dP}{dt} = 0.6e^{0.3t}$

d **i** At $t = 6$, $\frac{dP}{dt} = 0.6e^{0.3\times6} = 4$ waterbirds per month

ii At $t = 24$, $\frac{dP}{dt} = 0.6e^{0.3\times24} = 804$ waterbirds per month

12 $N = N_0\, e^{1.2t}$

a $N_0 = 30\,000$ bacteria

b $N_5 = 30\,000\, e^{1.2 \times 5} = 12\,102\,864$

c $\frac{dN}{dt} = 30\,000 \times 1.2e^{1.2t}$

i At $t = 5$ hours, $\frac{dN}{dt} = 30\,000 \times 1.2e^{1.2\times5} = 14\,523\,427$ bacteria per hour

ii At $t = 12$ hours, $\frac{dN}{dt} = 30\,000 \times 1.2e^{1.2\times12} = 6.459 \times 10^{10}$ bacteria per hour

iii At $t = 24$ hours, $\frac{dN}{dt} = 30\,000 \times 1.2e^{1.2\times24} = 1.159 \times 10^{17}$ bacteria per hour

Exercise 1.05 The derivatives of trigonometric functions

Concepts and techniques

1 **a** $\frac{d}{dx}[x + \sin(x)] = 1 + \cos(x)$

b $\frac{d}{dx}[6\sin(x)] = 6\cos(x)$

c $\frac{d}{dx}\sin(6x) = 6\cos(6x)$

d $\frac{d}{dx}\sin(x^2 - 3) = 2x\cos(x^2 - 3)$

e $\frac{d}{dx}4\sin\left(\frac{x}{3}\right) = \frac{4}{3}\cos\left(\frac{x}{3}\right)$

2 **a** $\frac{d}{dx}[\sin(x) + 9]^6 = 6[\sin(x) + 9]^5\cos x = 6\cos(x)[\sin(x) + 9]^5$

b $\frac{d}{dx}\left[x\sin(x)\right] = 1 \times \sin(x) + \left[\cos(x)\right]x = \sin(x) + x\cos(x)$

c $\frac{d}{dx}\sin(e^x) = e^x\cos(e^x)$

d $\frac{d}{dx}\left[\frac{\sin(2x)}{5x^2}\right] = \frac{2\cos(2x) \times 5x^2 - 10x\sin(2x)}{25x^4} = \frac{2x\cos(2x) - 2\sin(2x)}{5x^3}$

e $\frac{d}{dx}e^{\sin(x)} = \cos(x)e^{\sin(x)}$

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3 **a** $y = \sin(x)$

$$\frac{dy}{dx} = \cos(x)$$

At the point where $x = \frac{\pi}{3}$, $\frac{dy}{dx} = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

b $y = 3\sin(4x)$

$$\frac{dy}{dx} = 12\cos(4x)$$

At the point $\left(\frac{\pi}{16}, \frac{3}{\sqrt{2}}\right)$, $\frac{dy}{dx} = 12\cos\ 4\left(\frac{\pi}{16}\right) = 12\cos\left(\frac{\pi}{4}\right) = 12 \times \frac{1}{\sqrt{2}} = 6\sqrt{2}$

c $f(x) = \sin(x)$

$$f'(x) = \cos(x)$$

At the point $\left(\frac{3\pi}{2}, -1\right)$, $\frac{dy}{dx} = \cos\left(\frac{3\pi}{2}\right) = 0$

d $y = x + \sin(x)$

$$\frac{dy}{dx} = 1 + \cos(x)$$

At the point where $x = \frac{\pi}{6}$, $\frac{dy}{dx} = 1 + \cos\left(\frac{\pi}{6}\right) = 1 + \frac{\sqrt{3}}{2}$

e $y = \frac{\sin(x)}{x}$

$$\frac{dy}{dx} = \frac{x\cos(x) - 1 \times \sin(x)}{x^2} = \frac{x\cos(x) - \sin(x)}{x^2}$$

At the point where $x = \frac{\pi}{2}$, $\frac{dy}{dx} = \frac{0 - 1}{\left(\frac{\pi}{2}\right)^2} = -\frac{4}{\pi^2}$

4 **a** $\frac{d}{dx}[2 + \cos(x)] = -\sin(x)$

b $\frac{d}{dx}[3\cos(x)] = -3\sin(x)$

c $\frac{d}{dx}\cos(5x) = -5\sin(5x)$

d $\frac{d}{dx}\cos(3x^2 + 1) = -6x\sin(3x^2 + 1)$

e $\frac{d}{dx}2\cos\left(\frac{x}{2}\right) = -\sin\left(\frac{x}{2}\right)$

5 **a** $\frac{d}{dx}[4x + \cos(x)]^3 = 3[4x + \cos(x)]^2[4 - \sin(x)] = 3[4 - \sin(x)][4x + \cos(x)]^2$

b $\frac{d}{dx}[x\cos(x)] = \cos(x) - x\sin(x)$

c $\frac{d}{dx}\cos(e^{3x}) = -[\sin(e^{3x})]3e^{3x} = -3e^{3x}[\sin(e^{3x})]$

d $\frac{d}{dx}\left[\frac{\cos(x)}{3x}\right] = \frac{3x(-\sin x) - 3\cos(x)}{9x^2} = \frac{-x(\sin x) - \cos(x)}{3x^2}$

e $\frac{d}{dx}\cos[\sin(x)] = -\sin[\sin(x)] \times \cos(x)$

6 **a** $\frac{d}{dx}\tan(x) = \frac{1}{\cos^2(x)}$

b $\frac{d}{dx}[x + 6\tan(x)] = 1 + \frac{6}{\cos^2(x)}$

c $\frac{d}{dx}\tan(9x) = \frac{9}{\cos^2(9x)}$

d $\frac{d}{dx}3\tan(4x) = \frac{12}{\cos^2(4x)}$

e $\frac{d}{dx}[\tan(x) - 1]^5 = 5[\tan(x) - 1]^4 \times \frac{1}{\cos^2(x)} = \frac{5[\tan(x) - 1]^4}{\cos^2(x)}$

7 **a** $\frac{d}{dx}\, x^2 \tan(x) = 2x\tan(x) + \frac{x^2}{\cos^2(x)}$

b $\frac{d}{dx}\frac{\tan(2x)}{x} = \frac{\frac{2x}{\cos^2(2x)} - \tan(2x)}{x^2} = \frac{2}{x\cos^2(2x)} - \frac{\tan(2x)}{x^2}$

c $\frac{d}{dx} e^{\tan(x)} = \frac{e^{\tan(x)}}{\cos^2(x)}$

d $\frac{d}{dx}\tan[\cos(x)] = \frac{-\sin(x)}{\cos^2[\cos(x)]}$

e $\frac{d}{dx}\frac{\tan(x)}{e^x} = \frac{\frac{e^x}{\cos^2(x)} - e^x\tan(x)}{e^{2x}} = \frac{1}{e^x\cos^2(x)} - \frac{\tan(x)}{e^x}$

8 **a** $\frac{d}{dx}\, e^{2x}\sin(x) = 2e^{2x}\sin(x) + \left[e^{2x}\cos(x)\right] = e^{2x}\left[2\sin(x) + \cos(x)\right]$

At $x = 0.5$, $\frac{d}{dx}\, e^{2x}\sin(x) = e\left[2\sin(0.5) + \cos(0.5)\right]$

b $\frac{d}{dx}\left[x^2\tan(x)\right] = 2x\tan(x) + \frac{x^2}{\cos^2(x)}$

At $x = \frac{\pi}{4}$, $\frac{d}{dx}\left[x^2\tan(x)\right] = 2\left(\frac{\pi}{4}\right)\tan\left(\frac{\pi}{4}\right) + \frac{\left(\frac{\pi}{4}\right)^2}{\cos^2\left(\frac{\pi}{4}\right)} = \frac{\pi}{2} + \left(\frac{\pi}{4}\right)^2 \times 2 = \frac{\pi}{2} + \frac{\pi^2}{8}$

c $\frac{d}{dx}\cos^2(x) = 2\cos(x)[-\sin(x)]$

At $x = \frac{\pi}{3}$, $\frac{d}{dx}\cos^2(x) = 2\cos\left(\frac{\pi}{3}\right)\left[-\sin\left(\frac{\pi}{3}\right)\right] = 2 \times \frac{1}{2} \times \left(-\frac{\sqrt{3}}{2}\right) = -\frac{\sqrt{3}}{2}$

d $\frac{d}{dx}\frac{\cos(e^x)}{e^x} = \frac{-e^x\sin(e^x) \times e^x - e^x \times \cos(e^x)}{e^{2x}} = -\left[\frac{e^x\sin(e^x) + \cos(e^x)}{e^x}\right]$

At $x = 1$, $\frac{d}{dx}\frac{\cos(e^x)}{e^x} = -\left[\frac{e\sin(e) + \cos(e)}{e}\right]$

e $$\frac{d}{dx}\left[x^3\cos\left(x^2\right)\right] = 3x^2\cos\left(x^2\right) - 2x\left[\sin\left(x^2\right)\right]\times x^3$$

$$= x^2\left\{3\cos\left(x^2\right) - 2x^2\left[\sin\left(x^2\right)\right]\right\}$$

At $x = e$, $\frac{d}{dx}\left[x^3\cos\left(x^2\right)\right] = e^2\left\{3\cos\left(e^2\right) - 2e^2\left[\sin\left(e^2\right)\right]\right\}$

9 **a** $p(x) = x^2\sin(x) - x\cos(x)$

$p'(x) = [2x\sin(x) + x^2\cos(x)] - [\cos(x) - x\sin(x)]$

$p'(x) = 3x\sin(x) + x^2\cos(x) - \cos(x)$

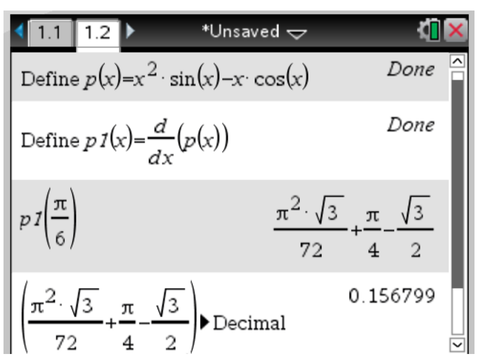
$$p'\left(\frac{\pi}{6}\right) = 3\times\left(\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) + \left(\frac{\pi}{6}\right)^2\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{6}\right)$$

$$= 3\times\left(\frac{\pi}{6}\right)\left(\frac{1}{2}\right) + \frac{\pi^2}{36}\times\left(\frac{\sqrt{3}}{2}\right) - \frac{\sqrt{3}}{2}$$

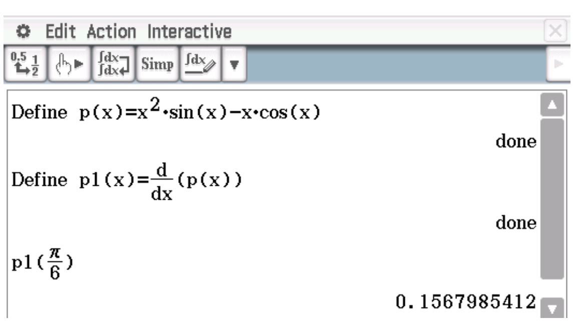
$$= \frac{\pi}{4} + \frac{\pi^2\sqrt{3}}{72} - \frac{\sqrt{3}}{2}$$

$$= 0.16 \quad (2\,\text{dp})$$

TI-Nspire CAS



ClassPad

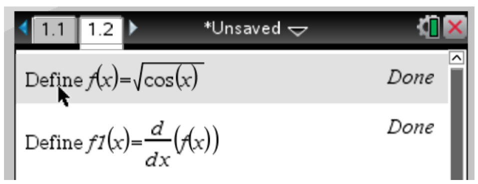


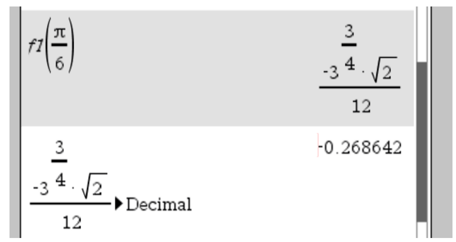
b $y = \sqrt{\cos(x)}$

$$\frac{dy}{dx} = \frac{1}{2}[\cos(x)]^{-\frac{1}{2}}[-\sin(x)] = \frac{-\sin(x)}{2\sqrt{\cos(x)}}$$

$$\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{6}} = \frac{-\sin\left(\frac{\pi}{6}\right)}{2\sqrt{\cos\left(\frac{\pi}{6}\right)}} = \frac{-\frac{1}{2}}{2\sqrt{\frac{\sqrt{3}}{2}}} = -0.269$$

TI-Nspire CAS





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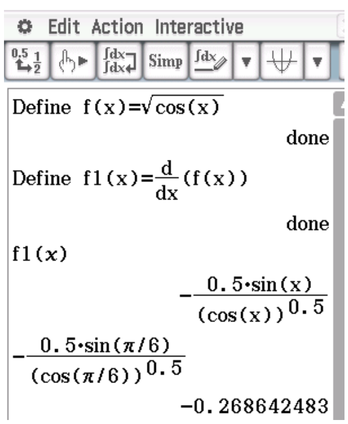

Reasoning and communication

10 $y = \tan(x) \Rightarrow \frac{dy}{dx} = \frac{1}{\cos^2(x)}$

$$\frac{1}{\cos^2(x)} = 2$$

$$\cos^2(x) = \frac{1}{2}$$

$$\cos(x) = \pm\frac{1}{\sqrt{2}}$$

$$x = \pm\frac{\pi}{4}, \pm\frac{3\pi}{4}, \pm\frac{5\pi}{4}, ...$$

$x = \pm\frac{\pi(2n+1)}{4}$, where n is any whole number.

11 **a** $x = 2\sin(3t)$

Greatest displacement from the centre is 2 cm.

b $x = 0$, so $2\sin(3t) = 0$

$3t = 0, \pi, 2\pi, 3\pi, ...$

$$t = 0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, ...$$

$t = \frac{n\pi}{3}$ for n any whole number

c $\frac{dx}{dt} = 6\cos(3t)$

At $t = \frac{\pi}{6}$ seconds, $\frac{dx}{dt} = 6\cos 3\left(\frac{\pi}{6}\right) = 0$ cm per sec

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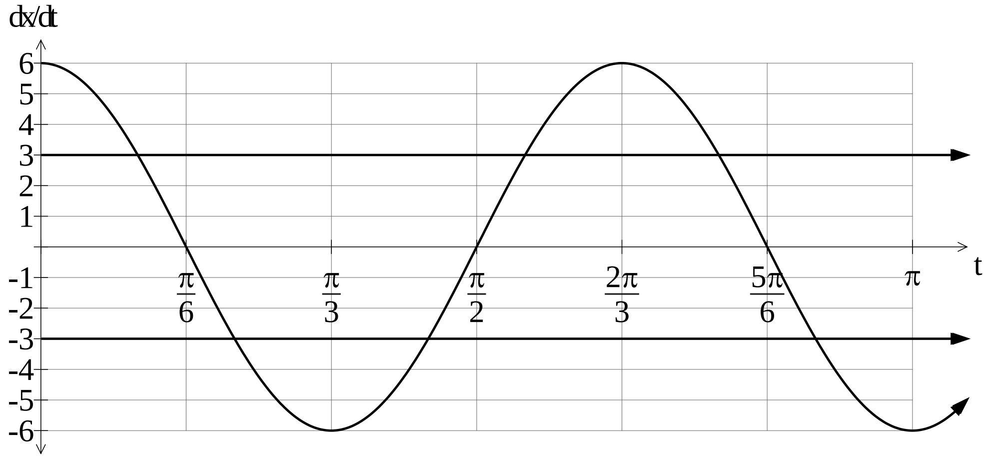
d $\frac{dx}{dt} = 6\cos(3t)$



$6\cos(3t) = \pm 3$, $t = ?$

$\cos(3t) = 0.5$ or $\cos(3t) = -0.5$

$3t = \frac{\pi}{3}, \frac{5\pi}{3}, ...$ or $3t = \frac{2\pi}{3}, \frac{4\pi}{3}, ...$

$t = \frac{\pi}{9}, \frac{5\pi}{9}, ...$ or $t = \frac{2\pi}{9}, \frac{4\pi}{9}, ...$

First three times are $\frac{\pi}{9}, \frac{2\pi}{9}, \frac{4\pi}{9}$

12 $V = \sin(2t) + 3t + 1$ mm/s.

a At $\frac{\pi}{12}$ seconds, $V = \sin\left(\frac{\pi}{6}\right) + 3\left(\frac{\pi}{12}\right) + 1 = 2.3$ mm/s.

b $a = \frac{dV}{dt} = 2\cos(2t) + 3$

At $t = \frac{\pi}{4}$ seconds, $\frac{dV}{dt} = 2\cos 2\left(\frac{\pi}{4}\right) + 3 = 0 + 3 = 3$ mm/s^2.

c The minimum value of $2\cos(2t)$ is –2.

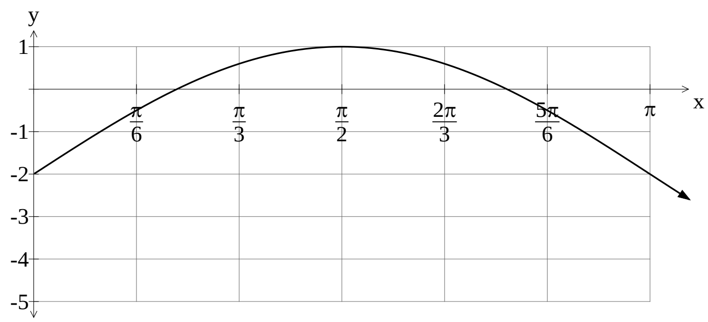
Therefore minimum acceleration is –2 + 3 = 1

Therefore the acceleration of the string is never 0.

Exercise 1.06 Applications of trigonometric functions and their derivatives

Reasoning and communication

1 $y = 3\sin(x) - 2$



If $y = 0$, then $\sin(x) = \frac{2}{3}$

$x = 0.72973$

$$\frac{dy}{dx} = 3\cos(x)$$

Gradient at $x = 0.72973$ is $3\cos(0.72973) = 2.23607$

$$\theta = \tan^{-1}(2.23607)$$

$$\theta = 1.15^c$$

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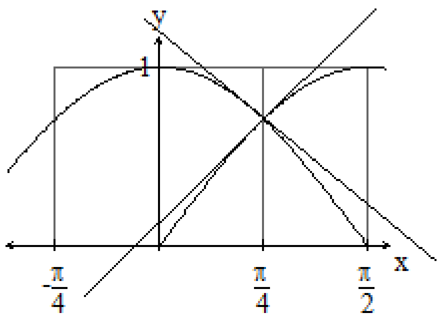
2 $\sin(x) = \cos(x)$ at $x = \frac{\pi}{4}$

$y = \sin(x)$ so $\frac{dy}{dx} = \cos(x)$



At $x = \frac{\pi}{4}$, the gradient of $y = \sin(x)$ is $\cos\left(\frac{\pi}{4}\right)$ i.e. $\frac{1}{\sqrt{2}}$

The angle made with the positive x-axis is $\theta = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) = 0.6155^c$

$y = \cos(x)$ so $\frac{dy}{dx} = -\sin(x)$

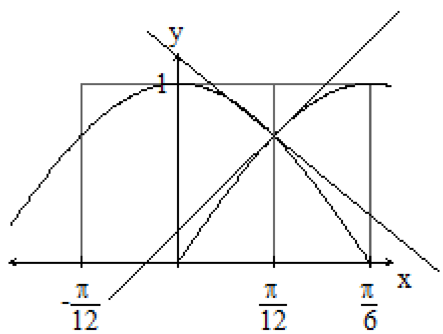
At $x = \frac{\pi}{4}$, the gradient of $y = \cos(x)$ is $-\sin\left(\frac{\pi}{4}\right)$ i.e. $-\frac{1}{\sqrt{2}}$

The angle made with the negative x-axis is $\theta = \tan^{-1}\left(-\frac{1}{\sqrt{2}}\right) = 0.6155^c$

The angle between the curves is $2 \times 0.6155^c = 1.23^c$ to 2 dp

3 sin (3x) = cos (3x) at tan (3x) = 1

$3x = \frac{\pi}{4}$, so $x = \frac{\pi}{12}$



y = sin (3x), so $\frac{dy}{dx} = 3\cos(3x)$

At $x = \frac{\pi}{12}$, the gradient of y = sin (3x) is 3 $\cos\left(\frac{\pi}{4}\right)$ i.e. $\frac{3}{\sqrt{2}}$

The angle made with the positive x-axis is $\theta = \tan^{-1}\left(\frac{3}{\sqrt{2}}\right) = 1.3803^c$

y = cos (3x), so $\frac{dy}{dx} = -3\sin(3x)$

At $x = \frac{\pi}{12}$, the gradient of y = cos (3x) is $-3\sin\left(\frac{\pi}{4}\right)$ i.e. $-\frac{3}{\sqrt{2}}$

The angle made with the negative x-axis is $\theta = \tan^{-1}\left(-\frac{3}{\sqrt{2}}\right) = 1.3803^c$

The angle between the curves is 2 × 1.3803^c = 2.26^c to 2 dp or the acute angle is 0.88^c.

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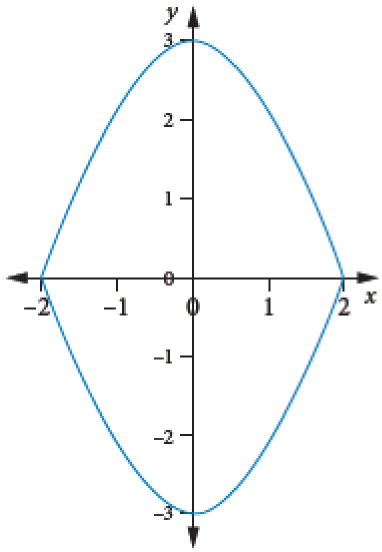
4 **a**



b The difference in radius between the *x* and *y* directions is 1 cm.

c $y = 3\cos\left(\frac{\pi x}{4}\right)$

$$\frac{dy}{dx} = -\frac{3\pi}{4}\sin\left(\frac{\pi x}{4}\right)$$

$$\text{At } x = -2,\ \frac{dy}{dx} = -\frac{3\pi}{4}\sin\left(\frac{-\pi}{2}\right) = \frac{3\pi}{4}$$

$$\tan(\theta) = \frac{3\pi}{4}$$

$$\theta = 1.17$$

so the angle is 2.34 radians (top and bottom)

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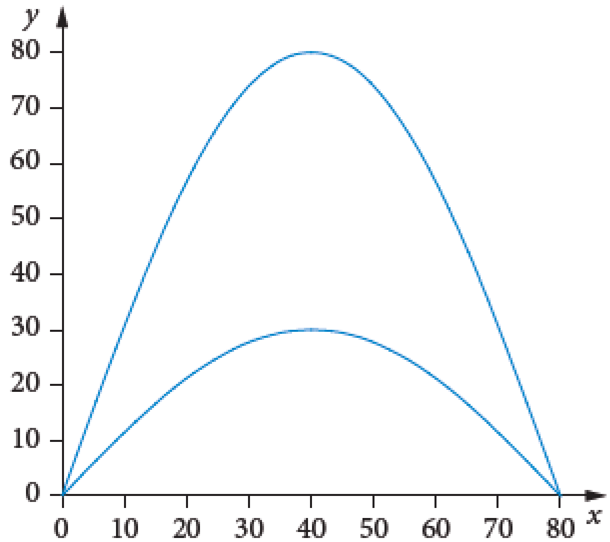
5 **a**



b 50 m

c $y = 80 \sin\left(\frac{\pi x}{80}\right)$, so $\frac{dy}{dx} = 80 \times \frac{\pi}{80}\cos\left(\frac{\pi x}{80}\right) = \pi\cos\left(\frac{\pi x}{80}\right)$

At $x = 0$, the gradient of $y = 80 \sin\left(\frac{\pi x}{80}\right)$ is $\pi\cos(0)$ i.e. π

The angle made with the positive x-axis is $\theta = \tan^{-1}(\pi) = 1.2626^c$

$y = 30 \sin\left(\frac{\pi x}{80}\right)$, so $\frac{dy}{dx} = 30 \times \frac{\pi}{80}\cos\left(\frac{\pi x}{80}\right) = \frac{3\pi}{8}\cos\left(\frac{\pi x}{80}\right)$

At $x = 0$, the gradient of $y = 30 \sin\left(\frac{\pi x}{80}\right)$ is $\frac{3\pi}{8}\cos(0)$ i.e. $\frac{3\pi}{8}$

The angle made with the positive x-axis is $\theta = \tan^{-1}\left(\frac{3\pi}{8}\right) = 0.8670^c$

The angle between the curves is $1.2626^c - 0.8670^c = 0.396^c$ to 3 dp.

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6 **a** $y = \sin(x) \Rightarrow \frac{dy}{dx} = \cos(x)$

At the point $\left(\frac{\pi}{6}, \frac{1}{2}\right)$, $\frac{dy}{dx} = \cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Equation of the tangent at $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ is $y = \frac{\sqrt{3}}{2}x + c$

Substitute $\left(\frac{\pi}{6}, \frac{1}{2}\right)$ to get c

$$\frac{1}{2} = \frac{\sqrt{3}}{2}\left(\frac{\pi}{6}\right) + c \Rightarrow c = \frac{1}{2} - \frac{\pi\sqrt{3}}{12}$$

$$y = \frac{\sqrt{3}}{2}x + \frac{1}{2} - \frac{\pi\sqrt{3}}{12}$$

b $y = -2\sin\left(\frac{x}{2}\right) \Rightarrow \frac{dy}{dx} = -\cos\left(\frac{x}{2}\right)$

Where $x = \frac{\pi}{2}$, $\frac{dy}{dx} = -\cos\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$

If $x = \frac{\pi}{2}$, then $y = \frac{-2}{\sqrt{2}} = -\sqrt{2}$

i.e. $\left(\frac{\pi}{2}, -\sqrt{2}\right)$

Equation of the tangent where $x = \frac{\pi}{2}$

$$y = -\frac{1}{\sqrt{2}}x + c$$

Substitute $\left(\frac{\pi}{2}, -\sqrt{2}\right)$ to get c

$$-\sqrt{2} = \left(-\frac{1}{\sqrt{2}}\right)\frac{\pi}{2} + c \Rightarrow c = -\sqrt{2} + \frac{\pi}{2\sqrt{2}}$$

$$y = -\frac{1}{\sqrt{2}}x - \sqrt{2} + \frac{\pi}{2\sqrt{2}}$$

c $y = \cos(x) \Rightarrow \frac{dy}{dx} = -\sin(x)$

At the point $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$, $\frac{dy}{dx} = -\sin\left(\frac{\pi}{6}\right) = -\frac{1}{2}$

Equation of the tangent at $\left(\frac{\pi}{6}, \frac{\sqrt{3}}{2}\right)$

$y - \frac{\sqrt{3}}{2} = -\frac{1}{2}\left(x - \frac{\pi}{6}\right)$

$12y - 6\sqrt{3} = -6x + \pi$

$6x + 12y - 6\sqrt{3} - \pi = 0$

d $y = \sin(2x) \Rightarrow \frac{dy}{dx} = 2\cos(2x)$

At the point $\left(\frac{\pi}{12}, \frac{1}{2}\right)$, $\frac{dy}{dx} = 2\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Equation of the tangent at $\left(\frac{\pi}{12}, \frac{1}{2}\right)$

$y - \frac{1}{2} = \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{12}\right)$

$24y - 12 = 12\sqrt{3}x - \sqrt{3}\pi$

$12\sqrt{3}x - 24y + 12 - \sqrt{3}\pi = 0$

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e $y = \tan(3x) \Rightarrow \frac{dy}{dx} = \frac{3}{\cos^2(3x)}$

At the point $\left(\frac{\pi}{12}, 1\right)$, $\frac{dy}{dx} = \frac{3}{\cos^2\left(\frac{\pi}{4}\right)} = 6$

Equation of the tangent at $\left(\frac{\pi}{12}, 1\right)$

$y = 6x + c$

Substitute $\left(\frac{\pi}{12}, 1\right)$ to get c

$1 = 6 \times \frac{\pi}{12} + c \Rightarrow c = 1 - \frac{\pi}{2}$

$y = 6x + 1 - \frac{\pi}{2}$

7 **a** $x = 3\cos\left(\frac{t}{2}\right) \Rightarrow v = -\frac{3}{2}\sin\left(\frac{t}{2}\right)$

b $a = -\frac{3}{4}\cos\left(\frac{t}{2}\right)$

c $0 = 3\cos\left(\frac{t}{2}\right)$

$\frac{t}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ ....

$t = \pi, 3\pi, 5\pi$......

d $v_{\pi} = -\frac{3}{2}\sin\left(\frac{\pi}{2}\right) = -\frac{3}{2}$

$v_{3\pi} = -\frac{3}{2}\sin\left(\frac{3\pi}{2}\right) = \frac{3}{2}$

$v_{5\pi} = -\frac{3}{2}\sin\left(\frac{5\pi}{2}\right) = -\frac{3}{2}$ and so on

$a_{\pi} = -\frac{3}{4}\cos\left(\frac{\pi}{2}\right) = 0$

$a_{3\pi} = -\frac{3}{4}\cos\left(\frac{3\pi}{2}\right) = 0$ and so on

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e $a = -\frac{3}{4}\cos\left(\frac{t}{2}\right)$

Greatest acceleration when $\cos\left(\frac{t}{2}\right) = -1$

i.e. $\frac{t}{2} = \pi, 3\pi, 5\pi, 7\pi....$

$\therefore t = 2\pi, 6\pi, 10\pi, 14\pi....$

8 **a** $v = 1.26 \sin(2\pi t)$

$a = 1.26 \times 2\pi \cos(2\pi t)$

$a = 2.52\pi \cos(2\pi t)$

b $v_5 = 1.26 \sin(2\pi \times 5) = 0$ m/s

$a_5 = 2.52\pi \cos(2\pi \times 5) = 7.92$ m/s^2

c If $v = -1.26$, $a = ?$

$v = -1.26$ when $2\pi t = 1.5\pi$, 3.5π, etc.

$a = 2.52\pi \cos(1.5\pi)$, etc $= 0$

$a = 0$

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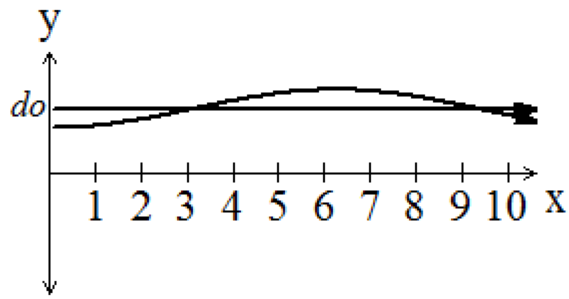
9 $d = d_0 - 0.9\cos(0.503t)$



Low tide at $t = 0$

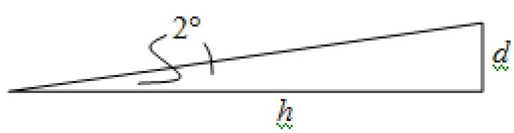
$$d = d_0 - 0.9\cos(0.503t)$$

$$\frac{dd}{dt} = 0 + 0.9 \times 0.503\sin(0.503t)$$

$$\frac{dd}{dt} = 0.4527\sin(0.503t)$$

At $t = 2$, $\frac{dd}{dt} \approx 0.38$ m/hr which is equivalent to 0.64 cm/min.

Mudflat slopes up at 2°.



$$\tan(2°) = \frac{h}{d}$$

$$h = \frac{d}{\tan(2°)}, \text{ so } \frac{dh}{dt} = \frac{1}{\tan(2°)}\frac{dd}{dt}$$

$$= \frac{0.4527\sin(0.503t)}{\tan(2°)} \approx 10.95 \text{ m/h} = \frac{100 \times 10.95}{60} \approx 18 \text{ cm/min}$$

10 **a** $x = 2\sin(3t)$

$v = 6\cos(3t)$

$a = -18\sin(3t)$

$= -9[2\sin(3t)]$

$\therefore\ a = -9x$

b $x = \mathrm{a}\cos(nt)$

$v = -\mathrm{a}n\sin(nt)$

$a = -\mathrm{a}n^2\cos(nt)$

$= -n^2[\mathrm{a}\cos(nt)]$

$\therefore\ a = -n^2 x$

Chapter 1 Review

Multiple choice

1 B $\frac{d}{dx}(3x^2 - 5x + 2)(x^4 + 3x^3 + x - 6)$

$= (6x - 5)(x^4 + 3x^3 + x - 6) + (4x^3 + 9x^2 + 1)(3x^2 - 5x + 2)$

$= 18x^5 + 20x^4 - 52x^3 + 27x^2 - 46x + 32$

2 B $\frac{d}{dx}\left(\frac{2x+1}{3x-2}\right) = \frac{2(3x-2)-3(2x+1)}{(3x-2)^2} = \frac{-7}{(3x-2)^2}$

3 D $\frac{d}{dx}\left(\frac{1}{x^4}\right) = -4x^{-5} = \frac{-4}{x^5}$

4 C $g(x) = (x^3 - 3x + 1)^3$

$g'(x) = 3(x^3 - 3x + 1)^2(3x^2 - 3) = 9(x^2 - 1)(x^3 - 3x + 1)^2$

$g'(-2) = 27$

5 E $y = e^{2x} \Rightarrow \frac{dy}{dx} = 2e^{2x}$

At $x = 1.5$, $\frac{dy}{dx} = 40.2$

6 D $y = 4\sin(e^x) + 2 \Rightarrow \frac{dy}{dx} = 4e^x\cos(e^x)$

$\left.\frac{dy}{dx}\right|_{x=2} = 4e^2\cos(e^2) = 13.25$

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7 B $y = \cos(3x) \Rightarrow \frac{dy}{dx} = -3\sin(3x)$

At the point where $x = \frac{\pi}{9}$, $\frac{dy}{dx} = -3\sin\left(\frac{\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ and $y = \cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$

Equation of tangent: $y = -\frac{3\sqrt{3}}{2}x + c$

Substitute the point $\left(\frac{\pi}{9}, \frac{1}{2}\right)$

$$\frac{1}{2} = -\frac{3\sqrt{3}}{2} \times \frac{\pi}{9} + c$$

$$c = \frac{1}{2} + \frac{\sqrt{3}\pi}{6}$$

$$y = -\frac{3\sqrt{3}}{2}x + \frac{1}{2} + \frac{\sqrt{3}\pi}{6}$$

$$6y = -9\sqrt{3}x + 3 + \sqrt{3}\pi$$

$$9\sqrt{3}x + 6y - 3 - \sqrt{3}\pi = 0$$

Short answer

8 **a** $\frac{d}{dx}x^5(3x^2 + 2x - 5) = 5x^4(3x^2 + 2x - 5) + x^5(6x + 2)$

$= 21x^6 + 12x^5 - 25x^4$

b $\frac{d}{dx}(x^2 + 1)(x^3 - 4x - 1) = 2x(x^3 - 4x - 1) + (3x^2 - 4)(x^2 + 1)$

$= 5x^4 - 9x^2 - 2x - 4$

9 **a** $\frac{d}{dx}\left(\frac{5x^3}{2x+1}\right) = \frac{15x^2(2x+1) - 2(5x^3)}{(2x+1)^2} = \frac{20x^3 + 15x^2}{(2x+1)^2}$

b $\frac{d}{dx}\left(\frac{x^2 + x - 2}{4x-3}\right) = \frac{(2x+1)(4x-3) - 4(x^2 + x - 2)}{(4x-3)^2} = \frac{4x^2 - 6x + 5}{(4x-3)^2}$

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10 **a** $\frac{d}{dx}x^{-6} = -6x^{-7} = -\frac{6}{x^7}$

b $\frac{d}{dx}\left(x^{\frac{1}{7}}\right) = \frac{1}{7}x^{\frac{-6}{7}} = \frac{1}{7x^{\frac{6}{7}}}$

c $\frac{d}{dx}\sqrt{x^3} = \frac{3}{2}x^{\frac{1}{2}} = \frac{3\sqrt{x}}{2}$

d $\frac{d}{dx}\left(\frac{9}{x^2}\right) = -18x^{-3} = \frac{-18}{x^3}$

11 **a** $\frac{d}{dx}(2x-7)^5 = 5 \times 2 \times (2x-7)^4 = 10(2x-7)^4$

b $\frac{d}{dx}3(2x^3 + x^2 - 3)^4 = 24x(2x^3 + x^2 - 3)^3(3x + 1)$

12 **a** $\frac{d}{dx}(x^5 + 1)^{-3} = -3(x^5 + 1)^{-4}(5x^4) = -15x^4(x^5 + 1)^{-4}$

b $\frac{d}{dx}\sqrt[3]{x-1} = \frac{1}{3}(x-1)^{\frac{-2}{3}} = \frac{1}{3\sqrt[3]{(x-1)^2}}$

13 **a** $\frac{d}{dx}[3x^2(x+2)^8]$

$= 6x(x + 2)^8 + 8(x + 2)^7 3x^2$

$= 3x(x + 2)^7[2(x+2) + 8x]$

$= 6x(5x + 2)(x + 2)^7$

b $\frac{d}{dx}\frac{3x^2}{(5x-1)^4} = \frac{6x(5x-1)^4 - 20(5x-1)^3 3x^2}{(5x-1)^8}$

$= \frac{(5x-1)^3\left[6x(5x-1)-60x^2\right]}{(5x-1)^8}$

$= \frac{\left(30x^2-6x-60x^2\right)}{(5x-1)^5}$

$= \frac{\left(-6x-30x^2\right)}{(5x-1)^5}$

$= \frac{-6x\left(1+5x\right)}{(5x-1)^5}$

14 **a** Given $y = e^x$, then $\frac{dy}{dx} = e^x$ at the point $(3, e^3)$ is e^3.

b $y = 3e^x \Rightarrow \frac{dy}{dx} = 3e^x$ so at the point where $x = -2$, $\frac{dy}{dx} = 3e^{-2}$

Equation of tangent $y = 3e^{-2}x + c$.

Substitute $(-2, 3e^{-2})$, $3e^{-2} = -6e^{-2} + c \Rightarrow c = 9e^{-2}$

Equation of tangent $y = 3e^{-2}x + 9e^{-2}$

or $3x - e^2y + 9 = 0$ (by multiplying by e^2)

15 **a** $\frac{d}{dx}\left(x^2 + 2e^x\right) = 2x + 2e^x$

b $\frac{d}{dx}e^{6x} = 6e^{6x}$

c $\frac{d}{dx}(e^x - 3)^9 = 9e^x(e^x - 3)^8$

d $\frac{d}{dx}2e^{4x+1} = 8e^{4x+1}$

e $\frac{d}{dx}(e^x + e^{-x}) = (e^x - e^{-x})$

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16 **a** $\frac{d}{dx}(4x+3)e^{2x} = 4e^{2x} + 2e^{2x}(4x+3) = 2e^{2x}(4x+5)$

b $\frac{d}{dx}\frac{e^{3x}}{x-4} = \frac{3e^{3x}(x-4)-1\times e^{3x}}{(x-4)^2} = \frac{e^{3x}(3x-13)}{(x-4)^2}$

17 **a** $N = 1000e^{0.24t} \Rightarrow N_0 = 1000e^0 = 1000$

b **i** $N_6 = 1000e^{0.24\times 6} = 4\ 221$

ii $N_{24} = 1000e^{0.24\times 24} = 317\ 348$

c $N = 1000e^{0.24t} \Rightarrow \frac{dN}{dt} = 1000\times 0.24e^{0.24t} = 240e^{0.24t}$

i At $t = 6$, $\frac{dN}{dt} = 240e^{0.24\times 6} = 1013$

ii At $t = 24$, $\frac{dN}{dt} = 240e^{0.24\times 24} = 76\ 164$

18 **a** **i** $T_2 = 75e^{-0.15\times 2} = 56°$

ii $T_5 = 75e^{-0.15\times 5} = 35°$

b $\frac{dN}{dt} = 75\times(-0.15)e^{-0.15t}$

$\frac{dN}{dt} = -11.25e^{-0.15t}$

i At $t = 2$ minutes,

$\frac{dN}{dt} = -11.25e^{-0.15\times 2} = -8.33°$ i.e. decreases at 8.33° per minute

ii At $t = 5$ minutes,

$\frac{dN}{dt} = -11.25e^{-0.15\times 5} = -5.31°$ i.e. decreases at 5.31° per minute

19 $\frac{dQ}{dt} = 0.04Q \Rightarrow Q = Q_0e^{0.04t}$

a The formula for the increase in salt over t weeks is given by $Q = 375e^{0.04t}$.

b At $t = 6$ weeks, $Q = 375e^{0.04\times 6} = 477$ grams

c $\frac{dQ}{dt} = 0.04Q \Rightarrow$ At $t = 6$ weeks, $\frac{dQ}{dt} = 0.04\times 477 = 19$ grams per week

20 **a** $\frac{d}{dx}[3\sin(x)+1] = 3\cos(x)$

b $\frac{d}{dx}[\sin(5x)] = 5\cos(5x)$

c $\frac{d}{dx} x\sin(2x) = \sin(2x) + 2x\cos(2x)$

d $\frac{d}{dx}\ [3x - \sin(x)]^7 = 7[3 - \cos(x)]\ [3x - \sin(x)]^6$

e $\frac{d}{dx}\sin(x^3+1) = 3x^2\cos(x^3+1)$

21 **a** $\frac{d}{dx}\cos\left(\frac{x}{3}\right) = -\frac{1}{3}\sin\left(\frac{x}{3}\right)$

b $\frac{d}{dx} e^x\cos(x) = e^x\cos(x) - e^x\sin(x) = e^x[\cos(x) - \sin(x)]$

c $\frac{d}{dx}\ [2+\cos(3x)]^5 = -15\sin(3x)[2+\cos(3x)]^4$

d $\frac{d}{dx}\cos(\pi x) = -\pi\sin(\pi x)$

e $\frac{d}{dx}\cos^2(x) = -2\sin(x)\cos(x)$

22 **a** $\frac{d}{dx} 6\tan(x) = \frac{6}{\cos^2(x)}$

b $\frac{d}{dx}\tan(3x) = \frac{3}{\cos^2(3x)}$

c $\frac{d}{dx}\tan\left(\frac{\pi x}{5}\right) = \frac{\pi}{5\cos^2\left(\frac{\pi x}{5}\right)}$

d $\frac{d}{dx} x^3\tan(2x) = 3x^2\tan(2x) + \frac{2x^3}{\cos^2(2x)}$

e $\frac{d}{dx}\frac{\tan(x)}{x} = \frac{\frac{x}{\cos^2(x)} - 1\times\tan(x)}{x^2} = \frac{1}{x\cos^2(x)} - \frac{\tan(x)}{x^2}$

Application

23 $Q = Q_0 e^{kt}$

$$\frac{dQ}{dt} = Q_0 \times k \times e^{kt}$$

$$= k\left(Q_0 e^{kt}\right)$$

$$\therefore \frac{dQ}{dt} = kQ$$

24 **a** $y = \tan(x)$

$$\frac{dy}{dx} = \frac{1}{\cos^2(x)}$$

At the point $\left(\frac{\pi}{4}, 1\right)$, $\frac{dy}{dx} = \frac{1}{\cos^2\left(\frac{\pi}{4}\right)} = 2$

Equation of the tangent $y = 2x + c$

$\left(\frac{\pi}{4}, 1\right)$, $y = 2x + c \Rightarrow 1 = 2\left(\frac{\pi}{4}\right) + c \Rightarrow c = 1 - \frac{\pi}{2}$

$y = 2x + 1 - \frac{\pi}{2}$

b $A\left(\frac{\pi}{4} - \frac{1}{2}, 0\right)$, $B\left(0, 1 - \frac{\pi}{2}\right)$

c $A_\Delta = \frac{1}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)\left(1 - \frac{\pi}{2}\right) = \frac{1}{16}(\pi - 2)(\pi - 2) = \frac{1}{16}(\pi - 2)^2 \approx 0.163 \text{ units}^2$

25 **a** $x = 6\sin(t) \Rightarrow x_5 = 6\sin(5) = –5.8$ cm

b if $x = 0$, $0 = 6\sin(t)$

$t = 0, \pi, 2\pi, 3\pi, \ldots$

i.e. $t = n\pi$, where n is any whole number.

c $v = 6\cos(t) \Rightarrow v_3 = 6\cos(3) = –5.94$ cm per sec

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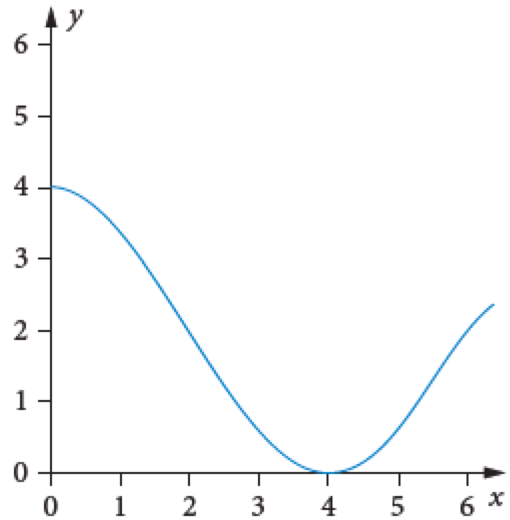
26 **a** $y = 2\cos\left(\frac{\pi x}{4}\right) + 2$



b $\frac{dy}{dx} = -2\times\frac{\pi}{4}\sin\left(\frac{\pi x}{4}\right)$

At $x = 6$, $\frac{dy}{dx} = -2\times\frac{\pi}{4}\sin\left(\frac{\pi(3)}{2}\right) = -\frac{\pi}{2}(-1) = \frac{\pi}{2}$

$\tan(\theta) = \frac{\pi}{2}$

$\theta = 57.52°$